Development of an analytical joint prior for effective spin distribution in GW astronomy

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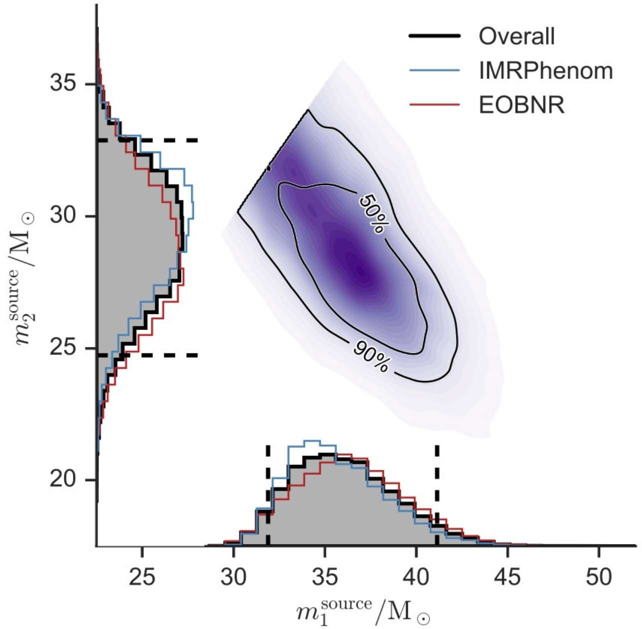
Single event inference of GW events

- For each GW event, the source parameters θ (masses, spins) are estimated by Bayesian inference
- GW provides the information of compact objects on the verge of merging, as well as remnant objects
- The statistical importance of parameter distribution of GW source is growing proportional to the expansion of GW catalog

Inferred component masses of GW150914
Figure from arXiv: 1602.03840



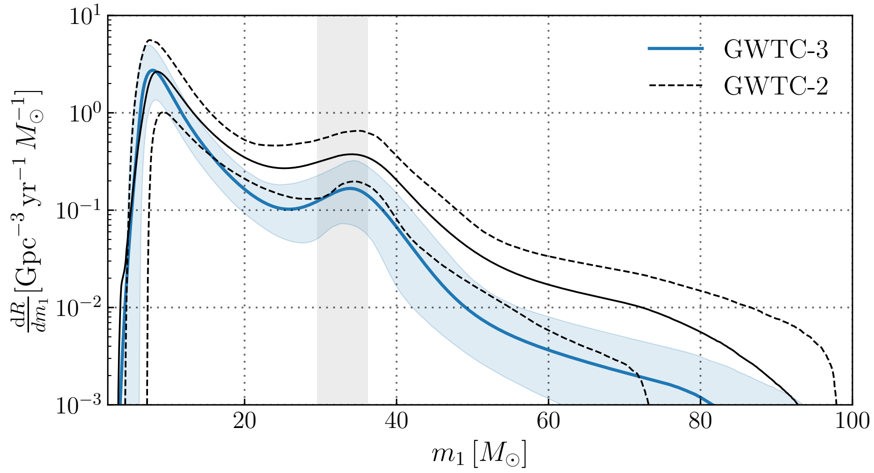
Population analysis in GW astronomy

- By combining each PE result, the distribution of source-parameter $p_{\mathrm{model}}(\theta)$ can be estimated → Population inference

- This method requires
  - The population model $p_{\mathrm{model}}(\theta|\Lambda)$
  - Set of astrophysical events $\{d\}$
  - Selection effect $\alpha(\Lambda)$

- Then, the hyper-likelihood can be obtained:

$$\mathcal{L}(\{d\}|\Lambda) \propto \prod_{i=1}^{N_{\mathrm{event}}} \frac{1}{\alpha(\Lambda)} \int d\theta_i \, p_{\mathrm{model}}(\theta_i|\Lambda) p(d_i|\theta_i)$$



Inferred m_1 distribution
$p_{\mathrm{model}}(m_1|\Lambda)$ = Power-law + Peak
Λ: set of population-level parameters (e.g., power-law index)

Figure from arXiv: 2111.03634

BH Spin and their astrophysical origin

Effect of BH spin on GW emerges at 1.5PN waveform

The spin of BBH provides clues of formation history of BBH

Inference on BH spin may help understanding mechanism of multi-messenger phenomena

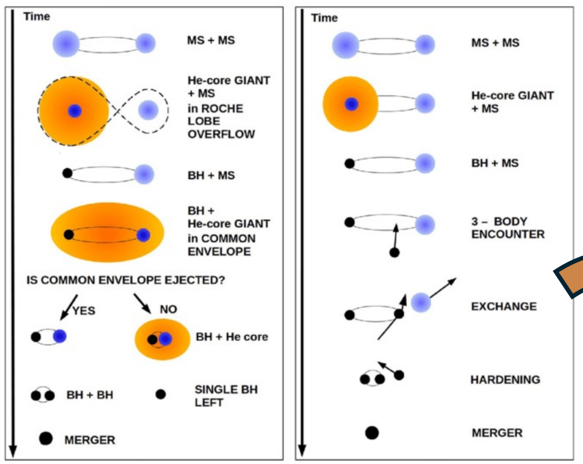
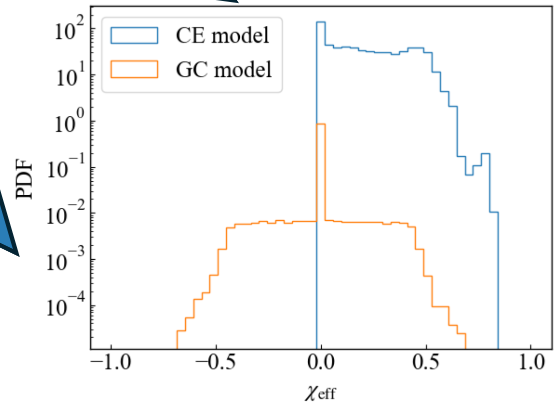


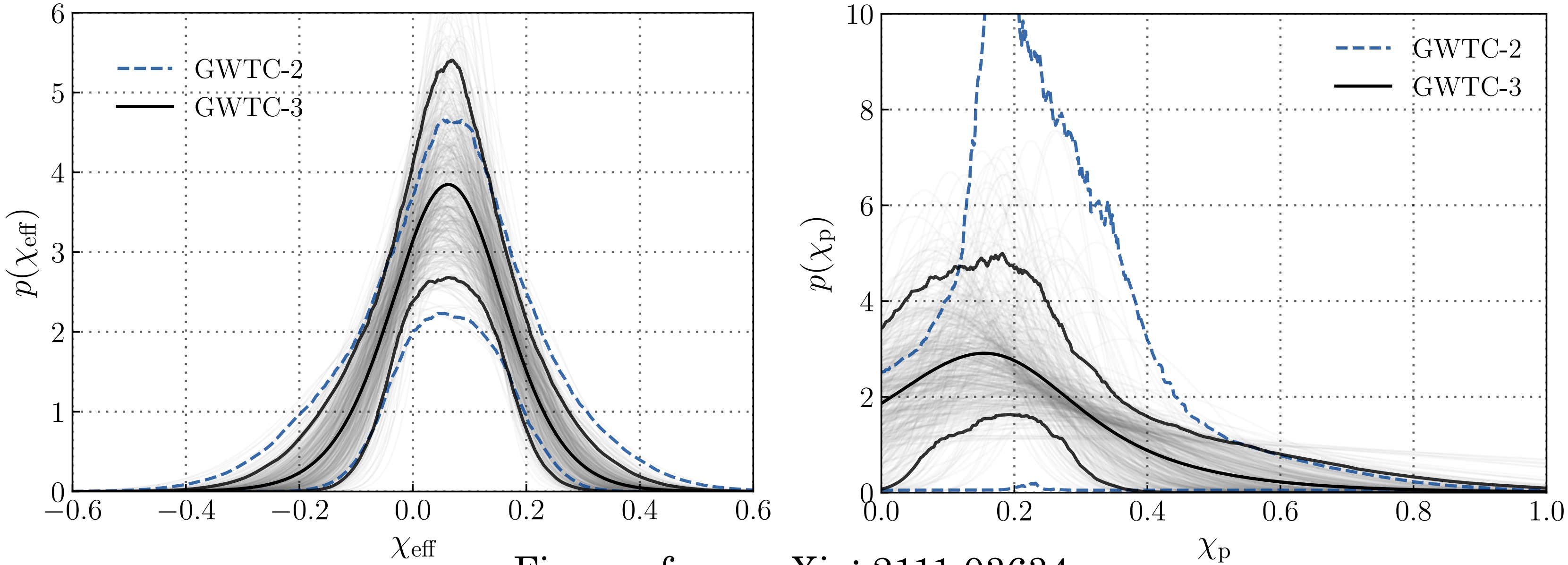
Figure from arXiv: 2105.12455

Data from arXiv: 2011.10057



GWTC-3 results on $\chi_{\rm eff}$, $\chi_{\rm p}$ distribution

$$p_{\rm model}(\chi_{\rm eff}, \chi_{\rm p} | \Lambda) \propto \exp\left[-\frac{1}{2(1-\rho^2)}\left[\left(\frac{\chi_{\rm eff}-\mu_{\rm eff}}{\sigma_{\rm eff}}\right)^2 - 2\rho\left(\frac{\chi_{\rm eff}-\mu_{\rm eff}}{\sigma_{\rm eff}}\right)\left(\frac{\chi_{\rm p}-\mu_{\rm p}}{\sigma_{\rm p}}\right) + \left(\frac{\chi_{\rm p}-\mu_{\rm p}}{\sigma_{\rm p}}\right)^2\right]\right]$$



Figures from arXiv: 2111.03634

Practical calculation of hyper-likelihood

- The hierarchical inference contains $N_{\rm event} + 1$ analytically impossible integrals

$$\mathcal{L}(\{d\}|\Lambda) \propto \prod_{i=1}^{N_{\rm event}} \frac{Z(d_i|\Lambda)}{\alpha(\Lambda)},$$

$$Z(d_i|\Lambda) = \int d\theta_i \, p_{\rm model}(\theta_i|\Lambda) p(d_i|\theta_i), \alpha(\Lambda) = \int d\theta \, p_{\rm model}(\theta|\Lambda) P_{\rm det}(\theta)$$

- These integrals are performed by Monte Carlo (MC) integrations
    - (theoretically) independent of the PE prior distribution
    - introduce MC uncertainty that can grow up to $O(N_{\rm event})$ into $p(\{d\}|\Lambda)$

The prior distribution and the MC sum

- MC integrations require the drawing distribution of posterior/injection samples:

$$Z(d_i|\Lambda) \propto \sum_j \frac{p_{\rm model}(\theta_i^j|\Lambda)}{\pi_{\rm PE}(\theta_i^j)}, \alpha(\Lambda) \propto \sum_j \frac{p_{\rm model}(\theta^j|\Lambda)}{\pi_{\rm inj}(\theta^j)}$$

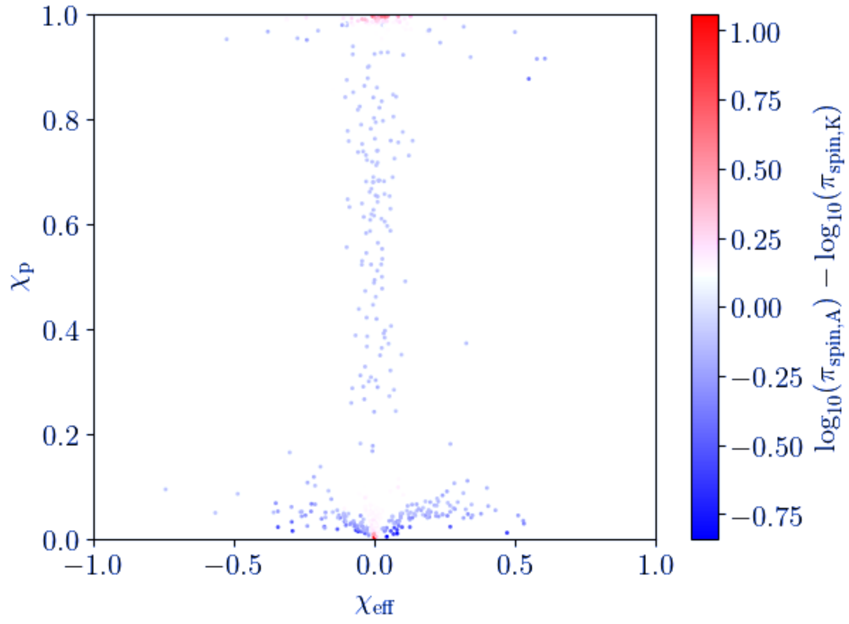
- Prior/injection distribution of spin parameters are $\pi_{\rm PE}(\vec{\chi}_i) = \pi_{\rm PE}(\chi_i, \cos\vartheta_i) \propto 1$
  - Uniform-in-amplitude and isotropic
- It must be reformulated in terms of $\chi_{\rm eff}, \chi_{\rm p}$ for models on $\chi_{\rm eff}, \chi_{\rm p}$:

$$\pi_{\rm spin}(\chi_{\rm eff}, \chi_{\rm p}|q) \coloneqq \int d\vec{\chi}_1 d\vec{\chi}_2\, \pi_{\rm PE}(\vec{\chi}_1)\pi_{\rm PE}(\vec{\chi}_2)\delta(\chi_{\rm eff} - \chi_{\rm eff}(\vec{\chi}_1, \vec{\chi}_2|q))\delta(\chi_{\rm p} - \chi_{\rm p}(\vec{\chi}_1, \vec{\chi}_2|q))$$

- In GWTC-2/3 analysis, this is performed in a numerical & stochastic way using Kernel density estimator (KDE)

Development of an analytical prior on $\chi_{\rm eff}, \chi_{\rm p}$

- We derive an analytical form of $\pi_{\rm spin}(\chi_{\rm eff}, \chi_{\rm p}|q)$
- This drastically decrease the computational time of $\pi_{\rm spin}(\chi_{\rm eff}, \chi_{\rm p}|q)$ and comes with zero uncertainty in estimating $\pi_{\rm spin}(\chi_{\rm eff}, \chi_{\rm p}|q)$
- The direct comparison of effective spin priors show that the KDE prior notably deviates from analytical formulation near the boundaries $\chi_{\rm p} \approx 0{,}1$ and $|\chi_{\rm eff}| \approx 0$ region



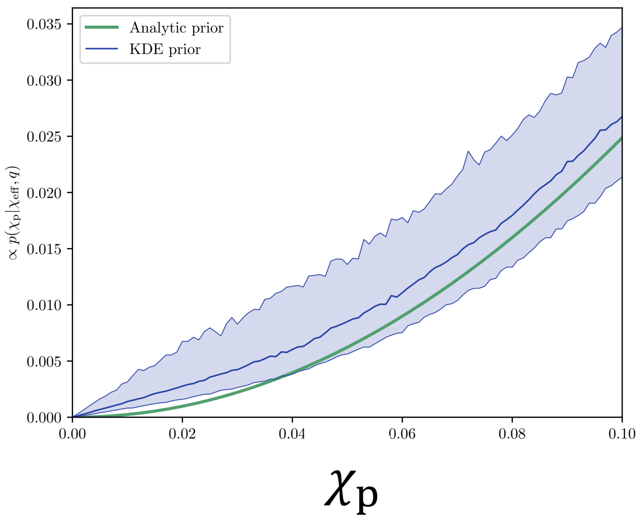
Differences between effective spin priors measured in logscale

KDE prior deviates when χ_p is small

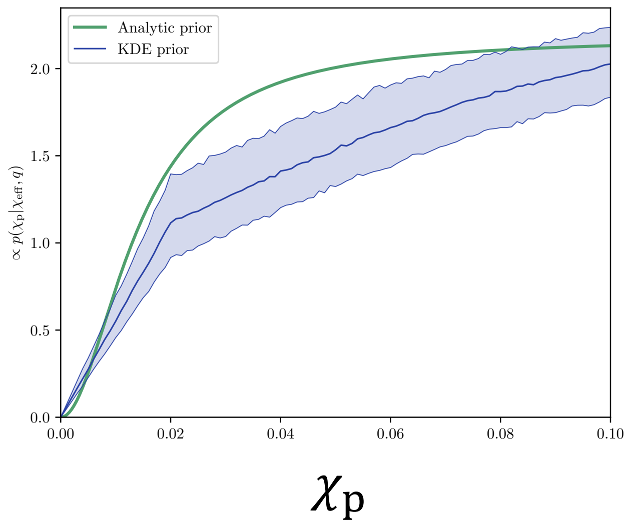
Near the boundary, the KDE prior suffers from the systematic bias

A boundary condition imposed on the KDE prior works well for relatively large $|\chi_{\rm eff}|$, but it is not suitable when both $|\chi_{\rm eff}|, \chi_p$ are small

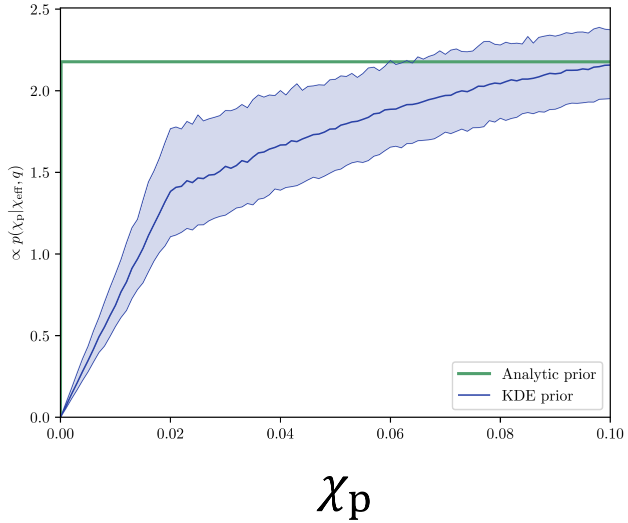
$\chi_{\rm eff} = 0.5$



$\chi_{\rm eff} = 0.01$



$\chi_{\rm eff} = 0$

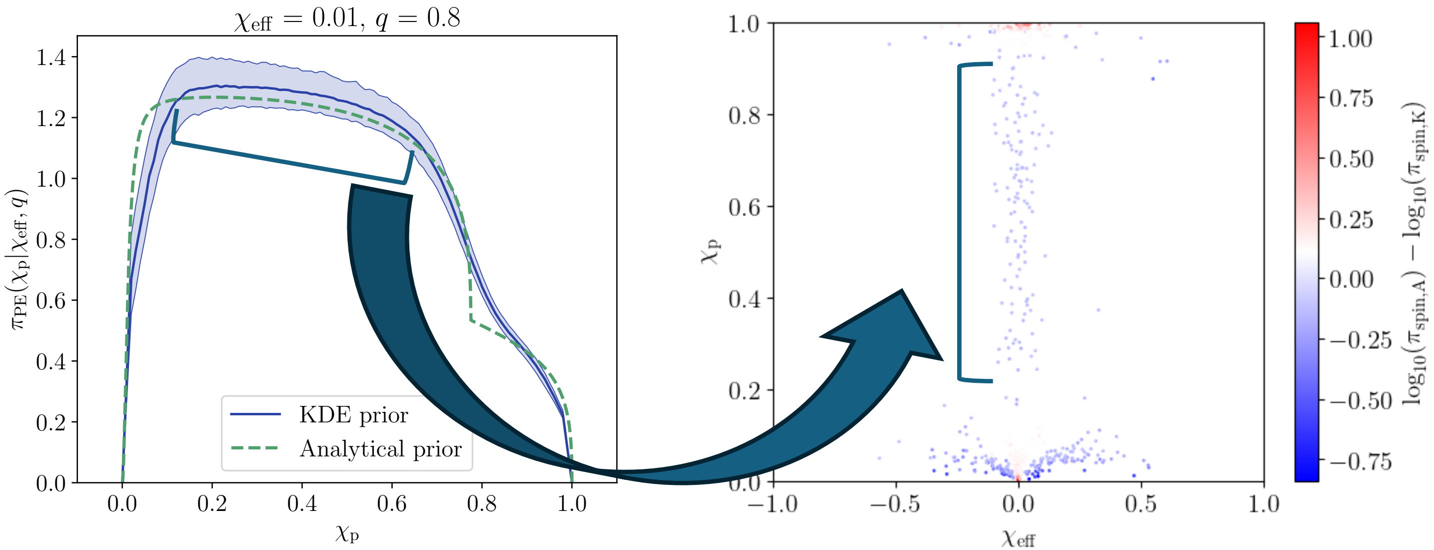


The normalization makes the KDE prior large

- The KDE prior has the normalization

$$\int d\chi_\mathrm{p}\, \pi_\mathrm{spin}(\chi_\mathrm{p}|\chi_\mathrm{eff}, q) = 1$$

- The value of $\pi_\mathrm{spin}(\chi_\mathrm{p}|\chi_\mathrm{eff} \approx 0, q)$ is overestimated because of the underestimation of $\pi_\mathrm{spin}(\chi_\mathrm{p} \approx 0|\chi_\mathrm{eff} \approx 0, q)$ and $\pi_\mathrm{spin}(\chi_\mathrm{p} \approx 1|\chi_\mathrm{eff} \approx 0, q)$



Setup of hierarchical Bayesian inference

- Re-analysis of GWTC-3 BBH spins with Gaussian spin model
- Consider 69 BBH events that are used in GWTC-3 analysis

$$\mathcal{L}(\{d\}|\Lambda) \propto \left[\sum_j \frac{p_{\mathrm{model}}(\theta^j|\Lambda)}{\pi_{\mathrm{inj}}(\theta^j)}\right]^{-N_{\mathrm{event}}} \prod_{i=1}^{N_{\mathrm{event}}} \left[\sum_j \frac{p_{\mathrm{model}}(\theta_i^j|\Lambda)}{\pi_{\mathrm{PE}}(\theta_i^j)}\right]$$

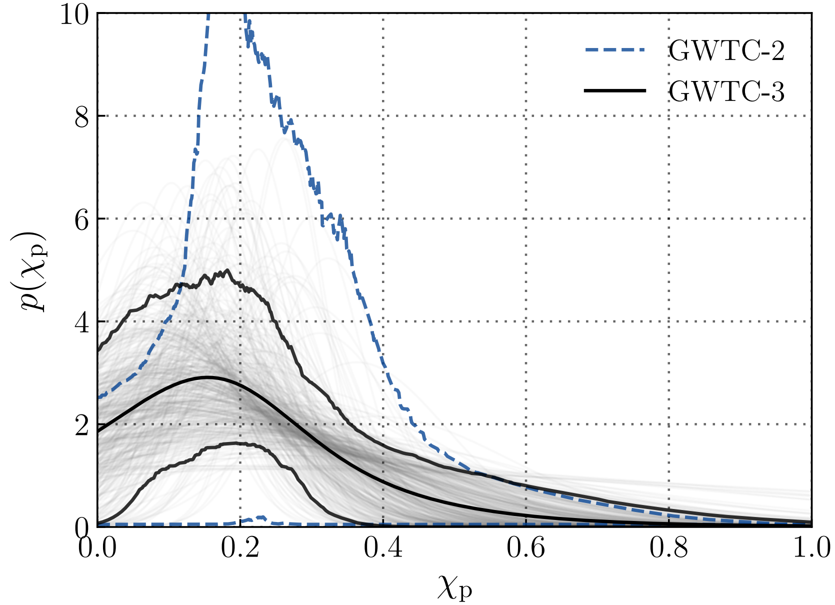
Changes from GWTC-3 analysis

- $\pi_{\mathrm{spin}}(\chi_{\mathrm{eff}}, \chi_{\mathrm{p}}|q)$ is updated with analytical form in hierarchical inference expression: $\pi_{\mathrm{inj}}(\theta^j)$ and $\pi_{\mathrm{PE}}(\theta_i^j)$
- All the posterior sample per event is used to minimize the MC integration error

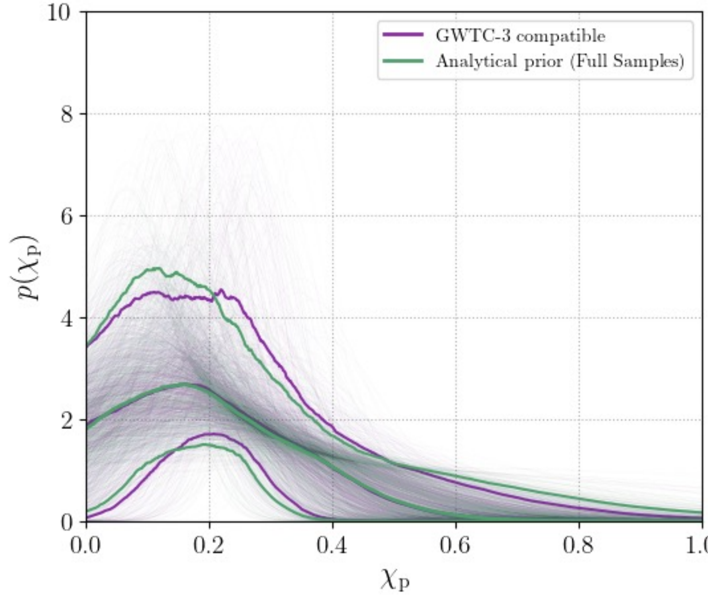
Recovered distribution of χ_p

- The posterior predictive distribution is consistent with the LVK's result

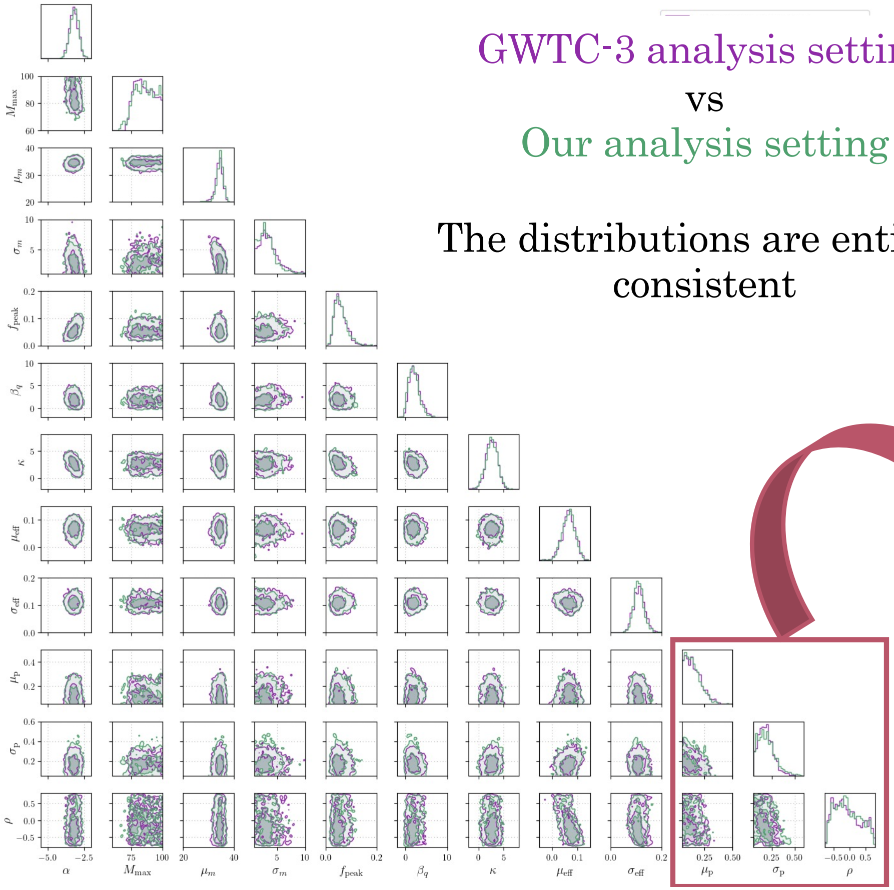
LVK's result
arXiv:2111.03634



Our run

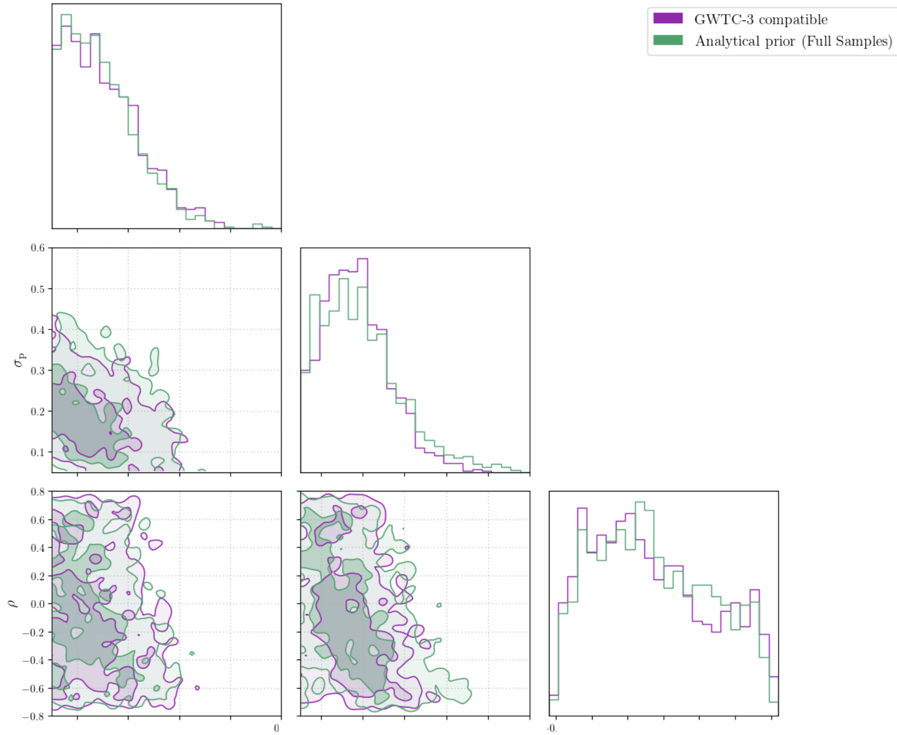


Result of rerun of hierarchical analysis



GWTC-3 analysis setting
vs
Our analysis setting

The distributions are entirely consistent



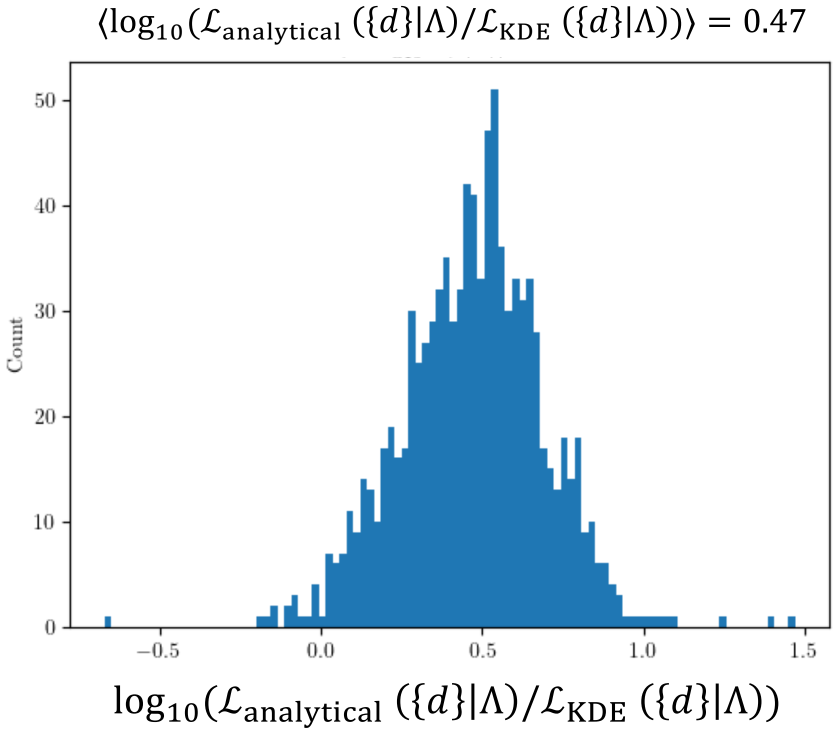
χ_p Gaussian Mean

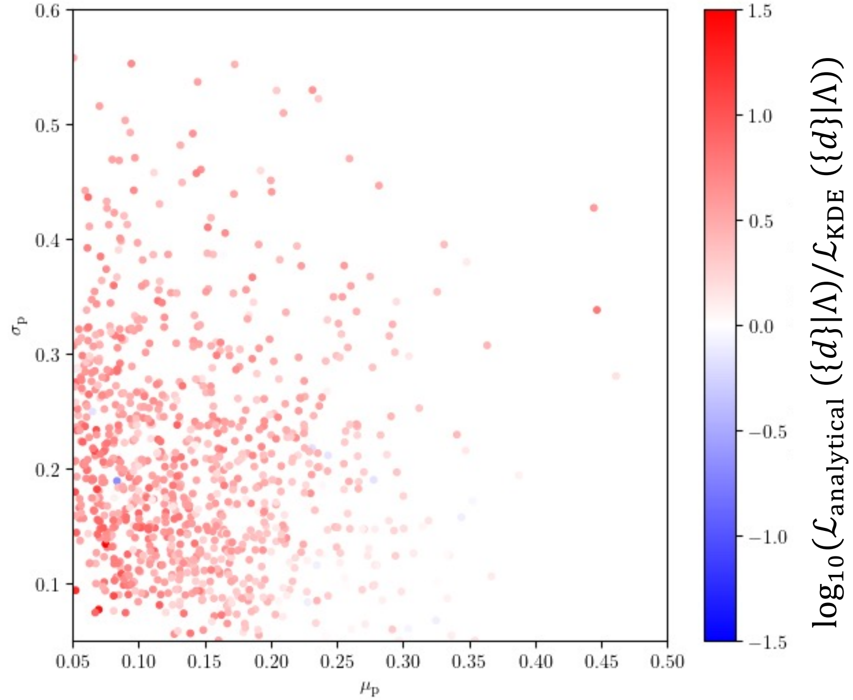
χ_p Gaussian SD

$\chi_{eff} - \chi_p$ Covariance

Differences in the likelihood evaluation

- The GWTC-3 analysis underestimates the hyper-likelihood $\mathcal{L}(\{d\}|\Lambda)$
- The introduction of analytical prior eliminates the systematic bias in $\mathcal{L}(\{d\}|\Lambda)$





Summary

- We derived an analytical form of the joint distribution of the effective spin parameters $\pi_{\text{spin}}(\chi_{\text{eff}}, \chi_{\text{p}}|q)$ which has been evaluated numerically
- Comparison with the KDE prior shows that the KDE prior deviates from $\pi_{\text{spin}}(\chi_{\text{eff}}, \chi_{\text{p}}|q)$ in some cases
- Re-analysis of GWTC-3 Gaussian Spin Model results in a widely consistent hyper-posterior distribution with KDE prior
  - but the introducing the analytical prior reveals the systematic bias on hyper-likelihood $\mathcal{L}(\{d\}|\Lambda)$