Robust 3+1D simulations of BDNK causal relativistic viscous hydrodynamics

Toward dissipative relativistic astrophysics

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VandyGRAF *(Vanderbilt initiative for gravity, waves, and fluids)*

arXiv:26??.xxxx Ongoing work with (Hirvonen H., Disconzi M., Bemfica F.S., Paquet JF, Noronha J.)

arXiv:2510.1660 (with Clarisse, N., Pinho, E. O., Patel, T., Bemfica, F. S., Hippert, M., & Noronha, J.)

U.S. DEPARTMENT of ENERGY
Office of Science

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Relativistic Hydrodynamics

Effective theory for long-wavelength fluid dynamics
Fluid variables: Velocity, Density and Pressure

Ideal (Inviscid) Fluids Non-dissipative

Non-Relativistic

$$\dot{\vec{u}} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla P$$

Relativistic

$$\nabla_\mu T^{\mu\nu} = 0 \qquad T_0^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$

4 DOFS

Energy-Momentum Tensor

Foundation of GRHD/GRMHD simulations in relativistic astrophysics

Ideal Relativistic Hydrodynamics + Einstein equations + Maxwell equations

Viscous Fluids Dissipative

Navier-Stokes

$$\dot{\vec{u}} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \vec{u}$$

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$

IDEAL

Viscous Terms

Relativistic Viscous Formulations

$$\dot{\vec{u}} + (\vec{u}\cdot\nabla)\vec{u} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\vec{u}$$

Recent interest in astrophysics
Most et al. '21
Chabanov & Rezzolla '23 BNS mergers
Espino et al. '23
Shibata et al. '25 Accretion

How do we formulate relativistic viscosity?

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$

IDEAL — Viscous Terms

Relativistic Generalization of Navier-Stokes

- Gradient Expansion: Eckart (1940) and Landau-Lifshitz (1950s).
- Acausal and Unstable.

Second-Order: Müller & Israel-Stewart (MIS) in the 60s.

- Viscous terms are variables with relaxation-type EOMs
- Causality not rigorously established.

(Bemifica et al., 2026 : Causality conditions for a class of MIS-type theories)

Heavy-ion collisions / Quark Gluon Plasma
Established application of relativistic viscous hydrodynamics

Causality + Stability + Hyperbolicity
Essential for relativity and numerical simulations

First-Order theories are always acausal and unstable?

BDNK: Causal First-order Viscous Hydrodynamics

Causal and Stable First-Order Gradient Expansion Theory

IDEAL $$T_0^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$

BDNK (**B**emfica, **D**isconzi, **N**oronha, **K**ovtun)

Bemfica, Disconzi and Noronha, 2018
Kovtun, 2019

Most general viscous energy-momentum tensor with first order gradient terms.

$$T_{\mu\nu} = (\varepsilon + \mathcal{A})u_\mu u_\nu + (P + \Pi)\Delta_{\mu\nu} - 2\eta\sigma_{\mu\nu} + u_\mu \mathcal{Q}_\nu + u_\nu \mathcal{Q}_\mu$$

Example: Gradient correction to the energy density $$\mathcal{A} = \tau_\varepsilon [u^\mu \nabla_\mu \varepsilon + (\varepsilon + P)\nabla_\mu u^\mu]$$

Explicit constraints on transport coefficients ⟺ CAUSAL | STABLE | HYPERBOLIC

Causality & Stability conditions: Can be checked when initializing
Can we turn it into a useful general numerical tool?

Simulating BDNK

2+1D Flat Spacetimes

Pandya et al. ’21, ‘22

2+1D QGP: Heavy Ion Collisions

Bantilan et al. 2022, Bea et al. 2023, Bhambure et al. 2025, Fantini et al. 2025

1+1D Neutron Stars

Keeble et al. 2025, Shumet et al. 2025.

Our Work : General 3+1 Dimensional Relativistic Simulations

2nd Order PDE System

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu} \quad \nabla_\mu T^{\mu\nu} = 0$$

1st Order flux conservative form (Finite volume methods)

$$\boxed{\partial_t \mathbf{Q} + \partial_i \mathbf{F_i} = \mathbf{S}}$$

Mathematical Formulation (2 Mathematically Equivalent Formulisms):

- Full First order formulism (Strictly Flux Conservative with Curl-Cleaning)
- Hybrid formulism (Computationally cheaper – Similar to HIC sims)

Numerical Implementation :

- Arbitrary Spacetimes (Thousands of terms)
- General Equation of State (Analytical and Tabulated)
- C++ Portable code kokkos (Scalable on CPUs and GPU clusters)

Testing is crucial!

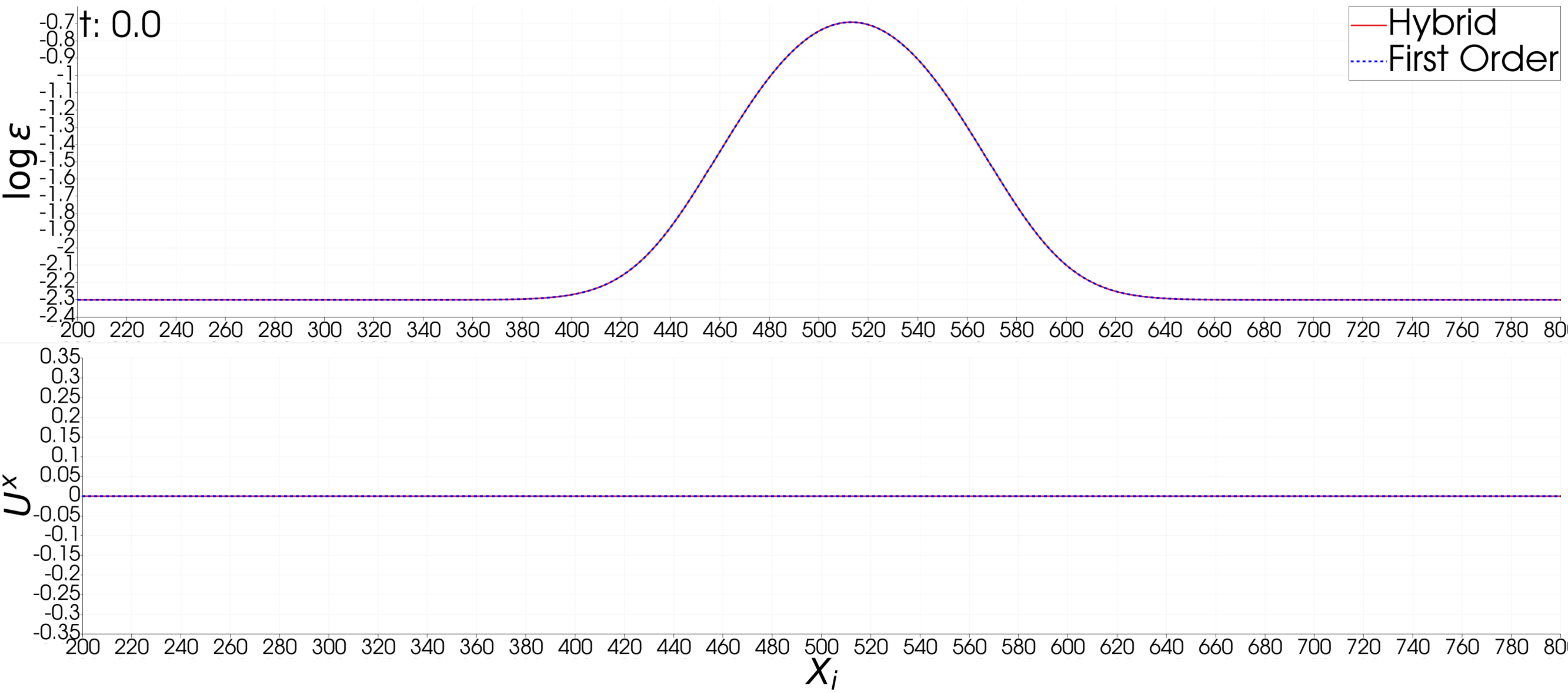
Convergence tests

Shock tests

Low viscosity ⟷ Ideal Hydro

Validation I (2 Formulisms): 1+1 Gaussian

Comparing the results from the two independent formalisms



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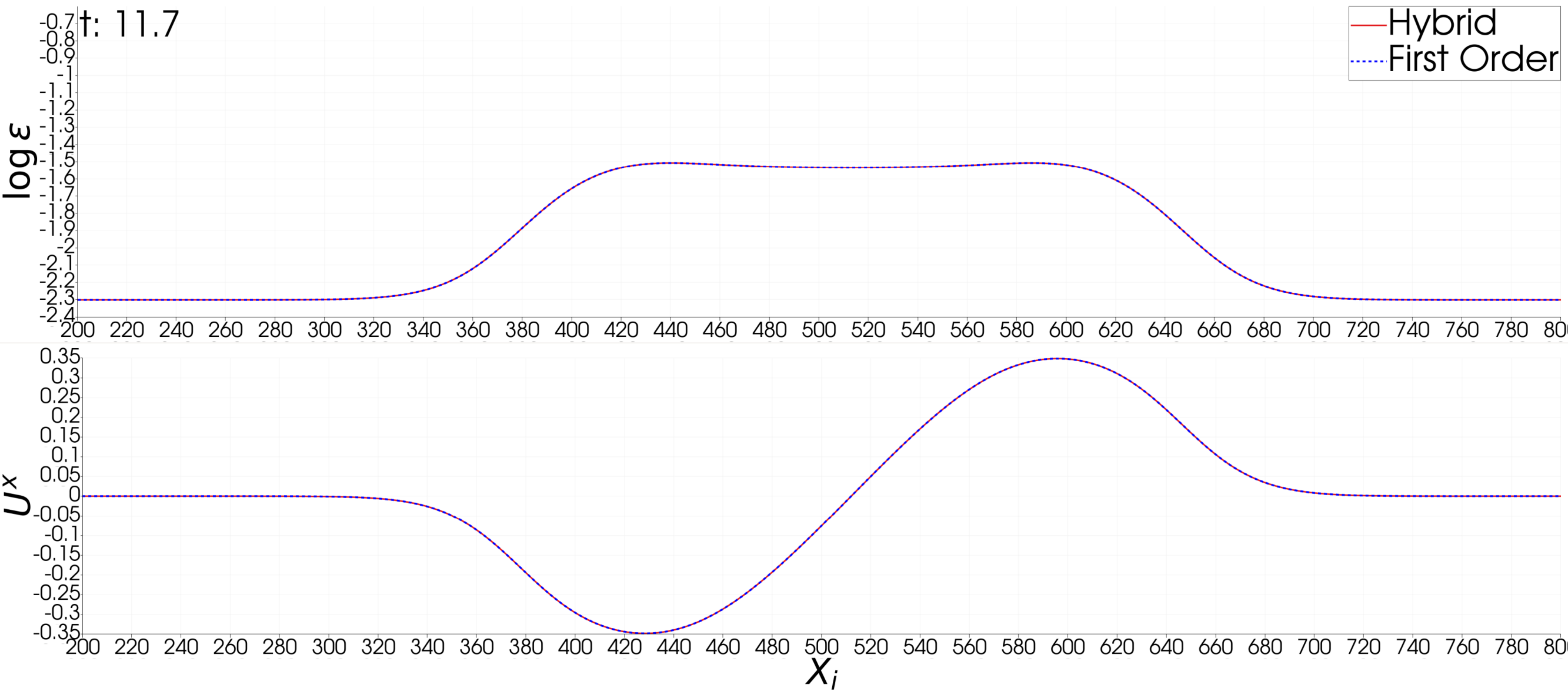

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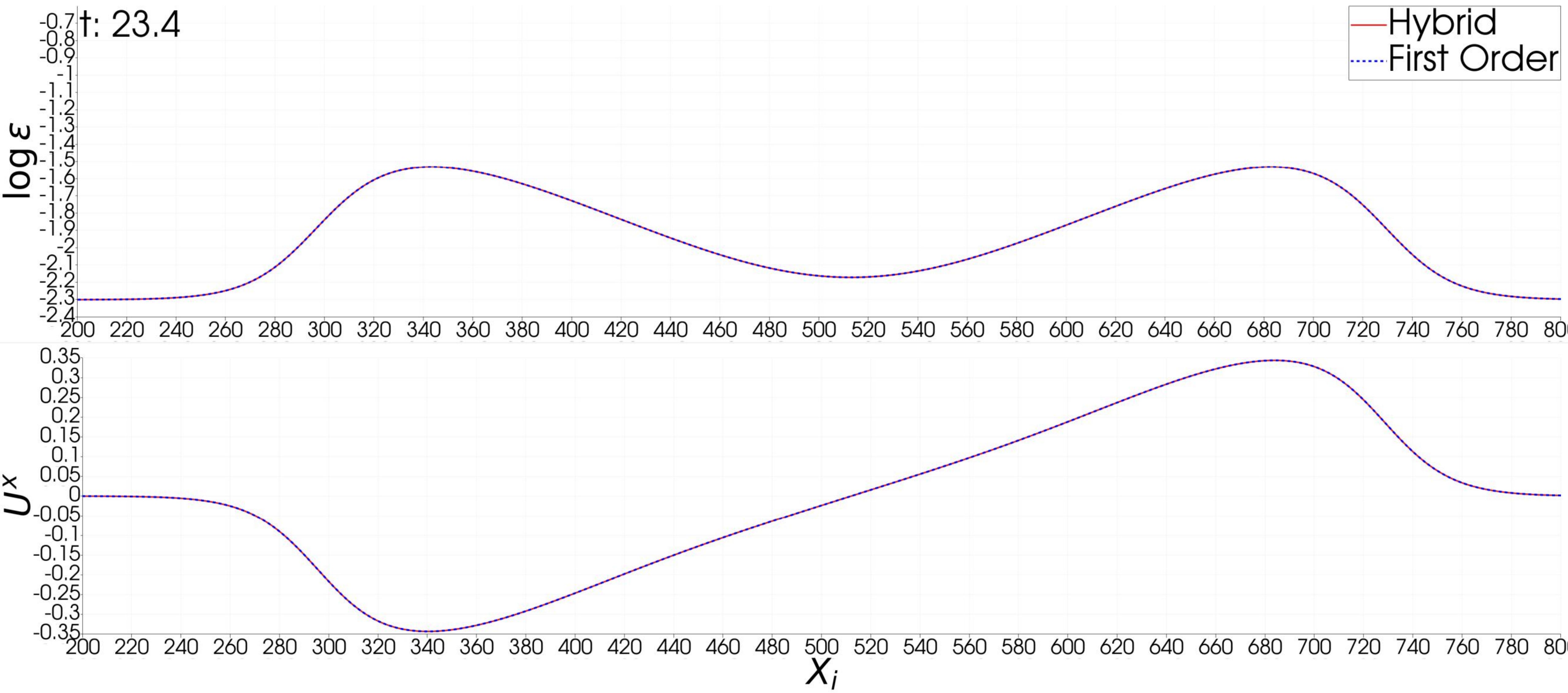
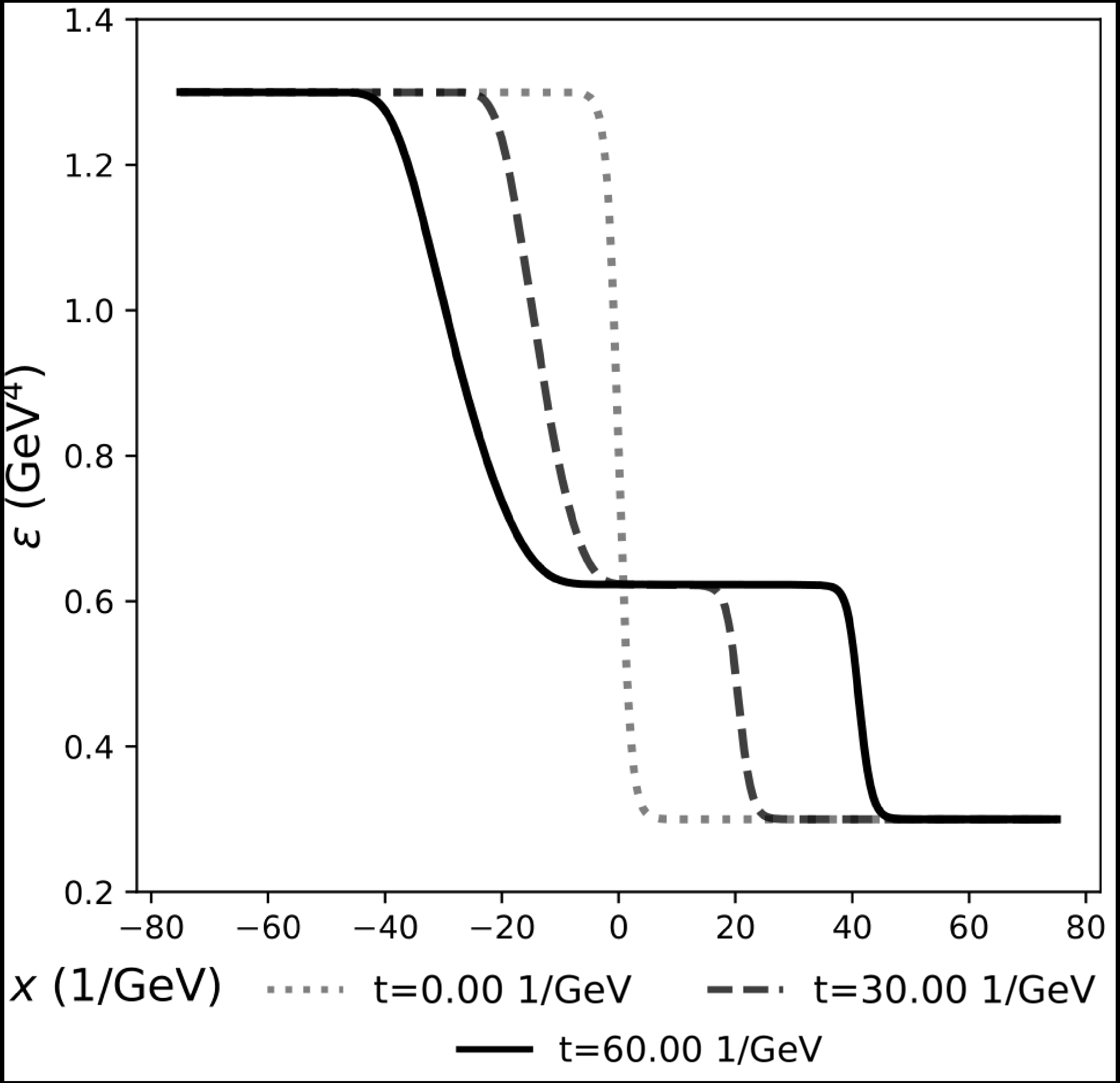


Validation II: Shock Tube tests

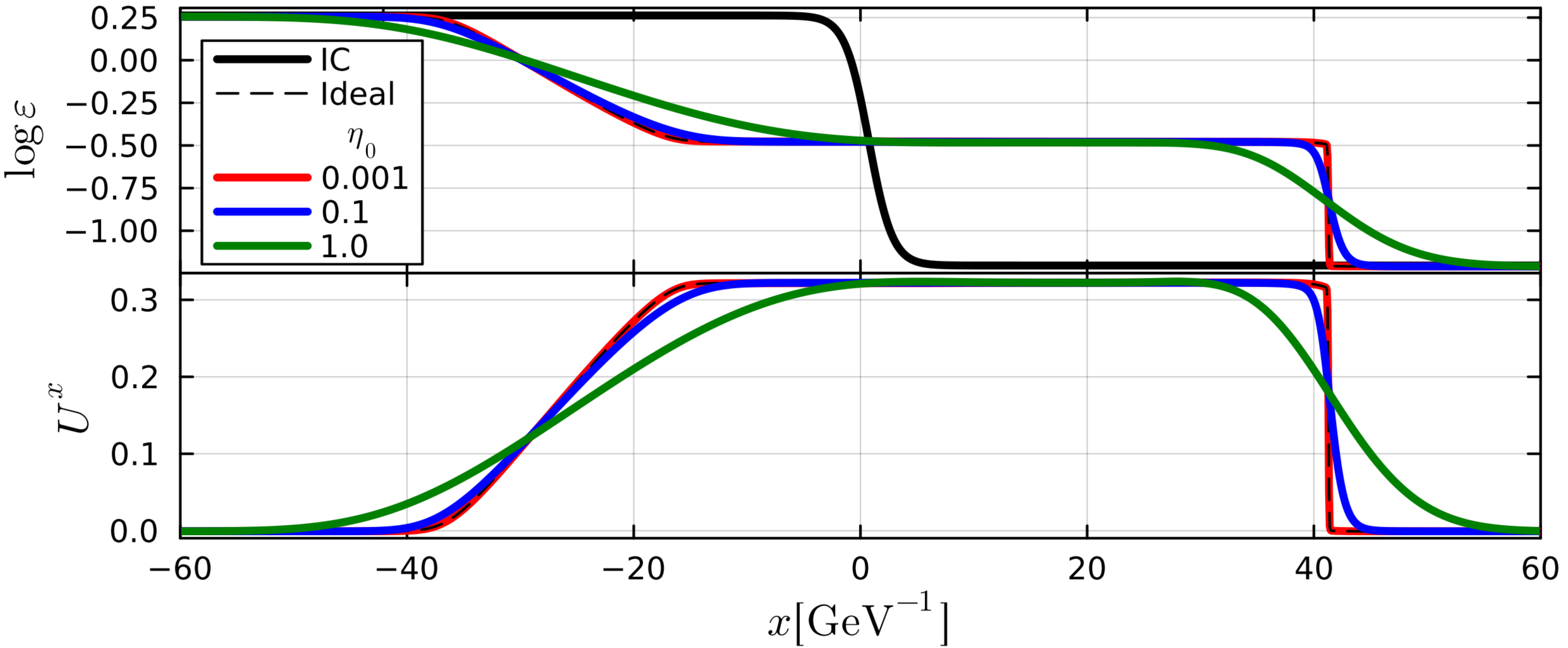
Shock tests conducted to check numerical stability : With an independent code

Clarisse, N., Pinho, E. O., Patel, T., Bemfica, F. S., Hippert, M., & Noronha, J. (2025). *arXiv:2510.16603*.

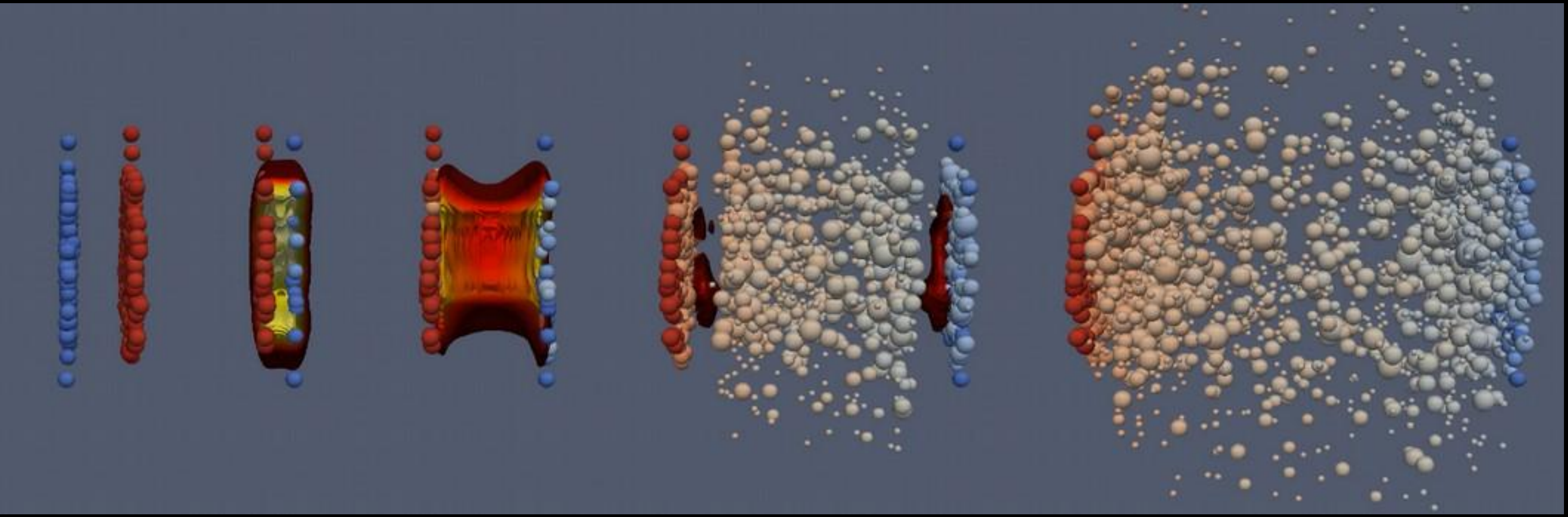


Validation II: Shock Tube tests

- Validated against code of Clarisse et al (2025)
- Agreement with Ideal at low viscosity



Validation III: Gubser Flow



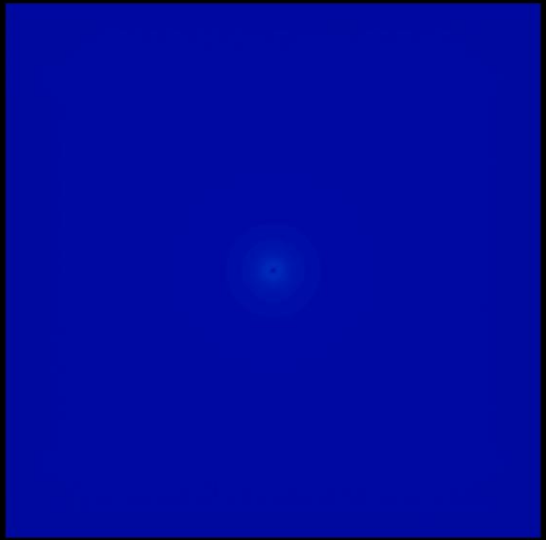


Illustration of a Heavy Ion Collision [Ref: MADAI collaboration, Hannah Petersen and Jonah Bernhard]

Hyperbolic coordinates used in heavy ion collisions

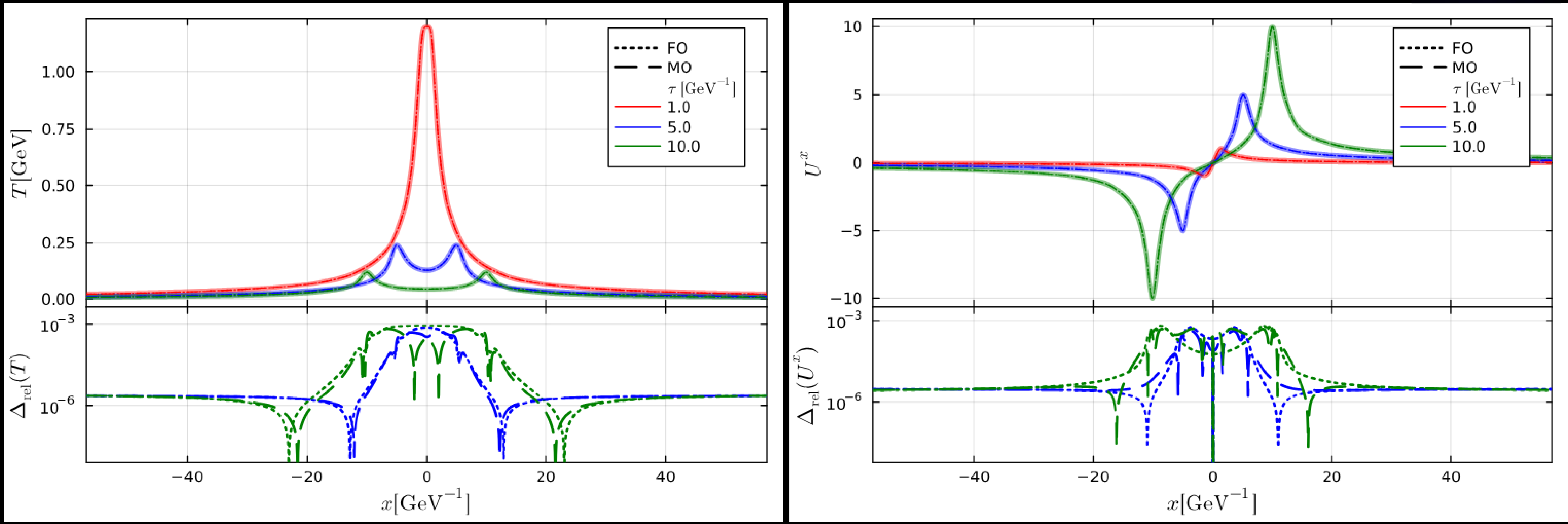
Minkowski metric with curvilinear coordinates

$$(\tau = \sqrt{t^2 - z^2}, x, y, \eta = \text{arctanh}(z/t))$$

Gubser Flow

- Cylindrically symmetric flow with semi-analytical solution
  - Toy model for QGP expansion
  - Semi-analytical solution : Bemfica et al. 2018

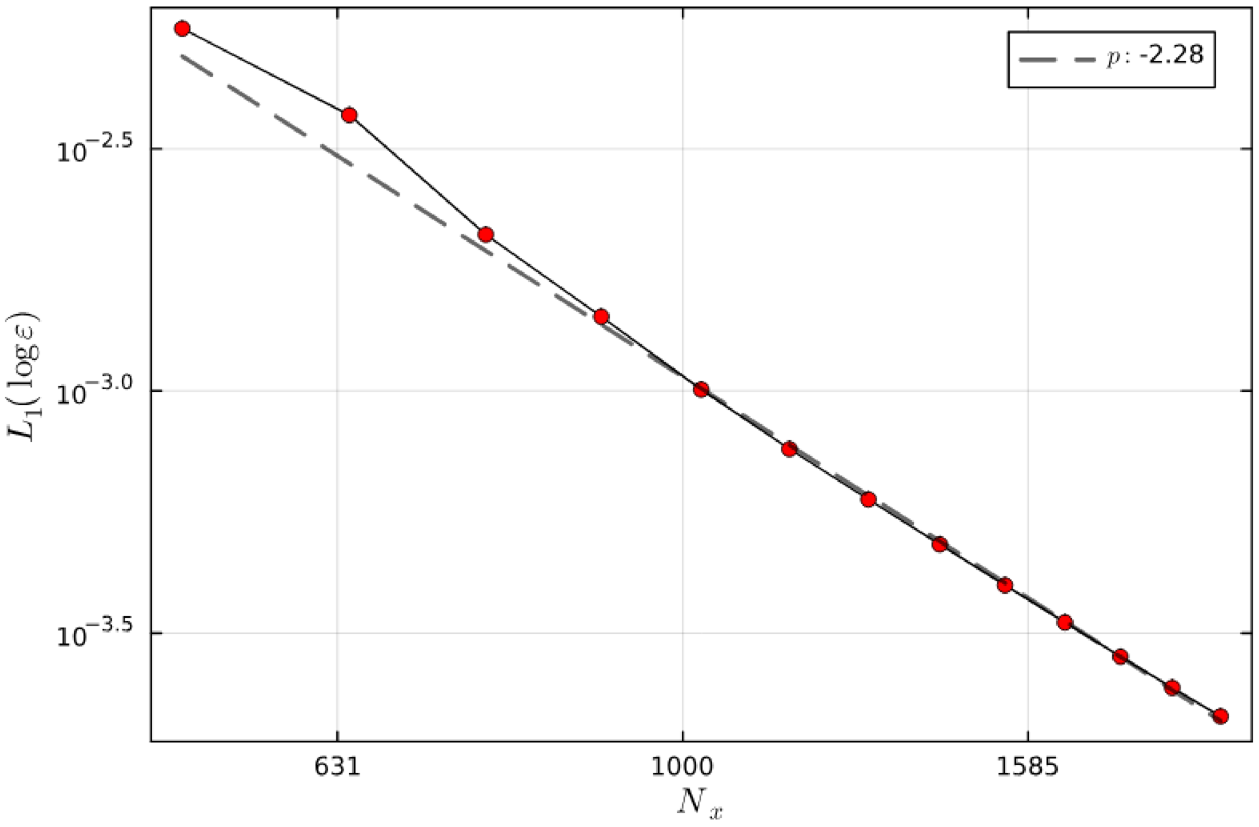
Validation III: Gubser

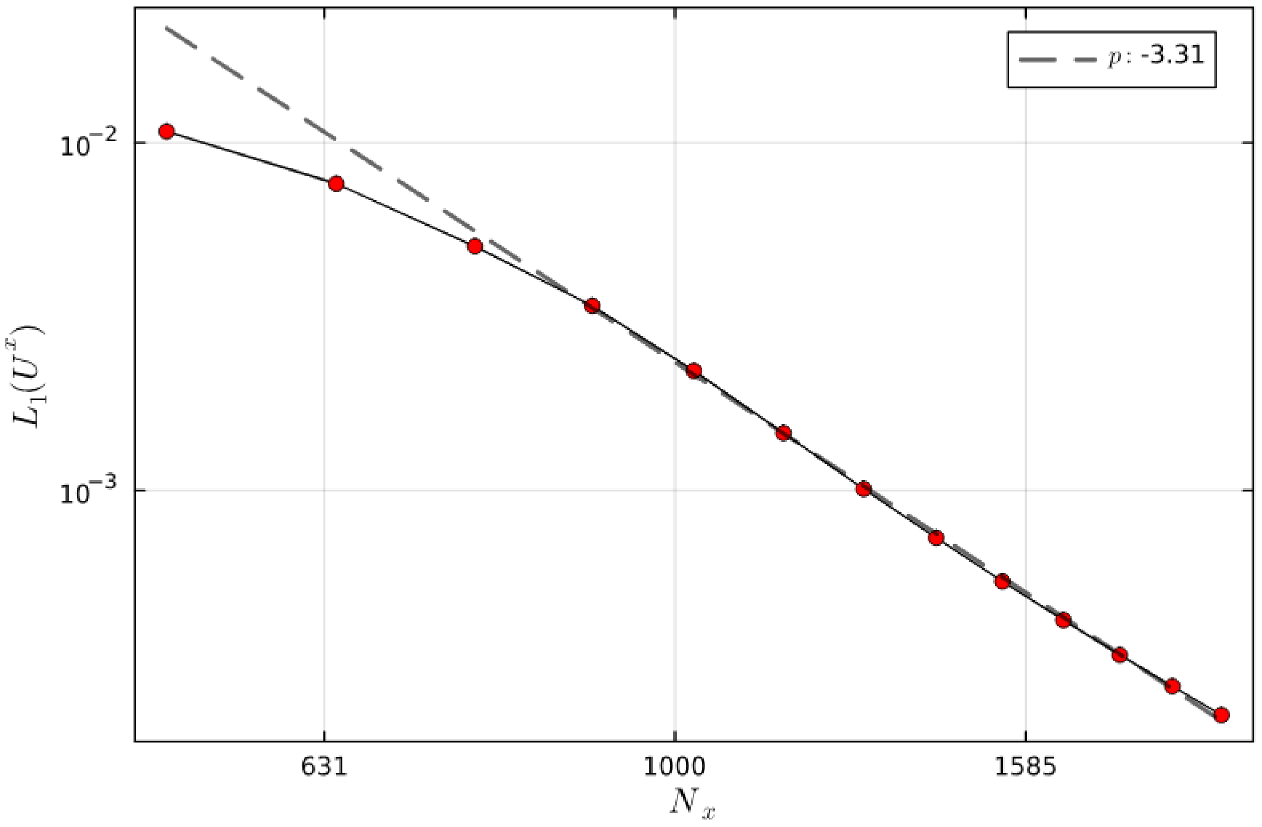


- Conformal fluid set up in 2+1D (xy-transverse plane).
- Initialized with semi-analytical solution for temperature and velocity.
- Agreement with semi-analytical solution.
- Two formulisms are consistent.

Validation III: Gubser

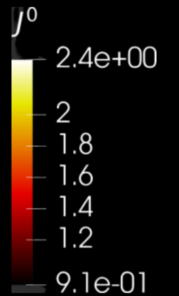
Convergence Test : Simulation should converge to the continuum solution with resolution *at a certain rate*





- Error scaling for 2nd Order Finite Volume methods $\quad \mathrm{Error} \propto \Delta x^{-2}$
- Error computed relative to the true semi-analytical solutions

Validation IV (Ideal Recovery): Kelvin Helmholtz

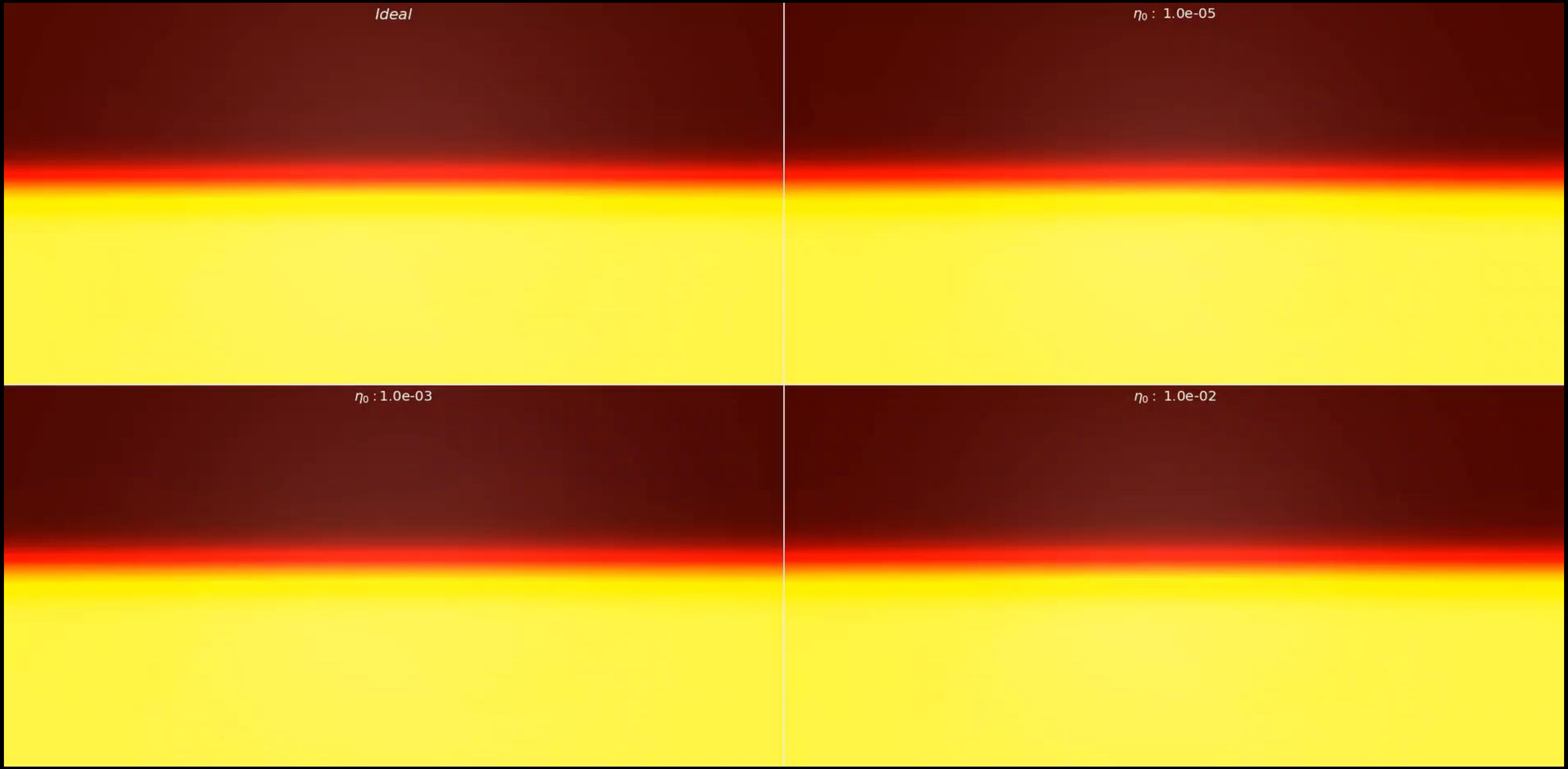


Low viscosity BDNK matches Ideal Hydro solution

$\eta_0 = 0$ $\eta_0 = 10^{-5}$

Inviscid

Smallest viscosity



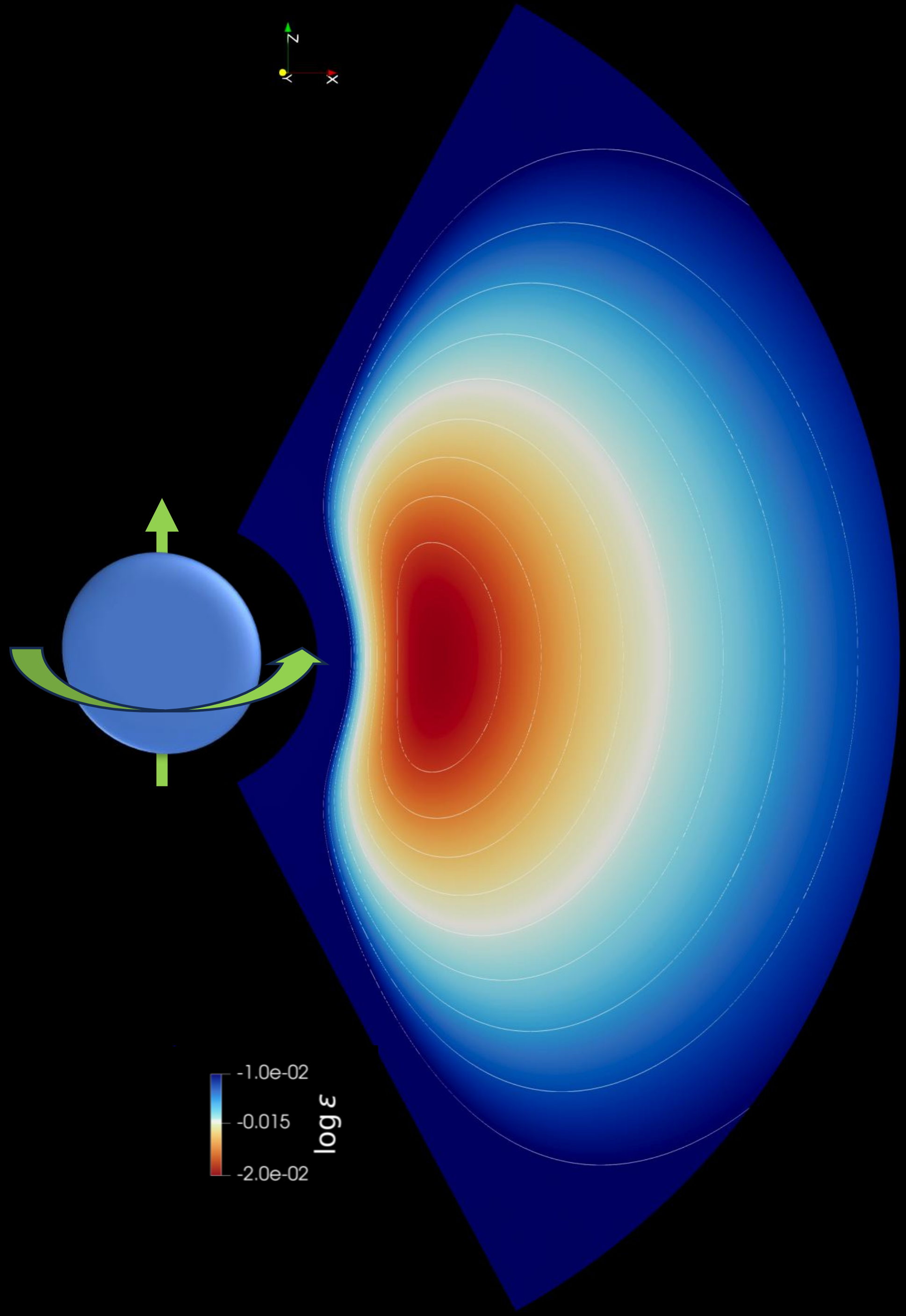
Intermediate viscosity

Largest viscosity

$\eta_0 = 10^{-3}$ $\eta_0 = 10^{-2}$

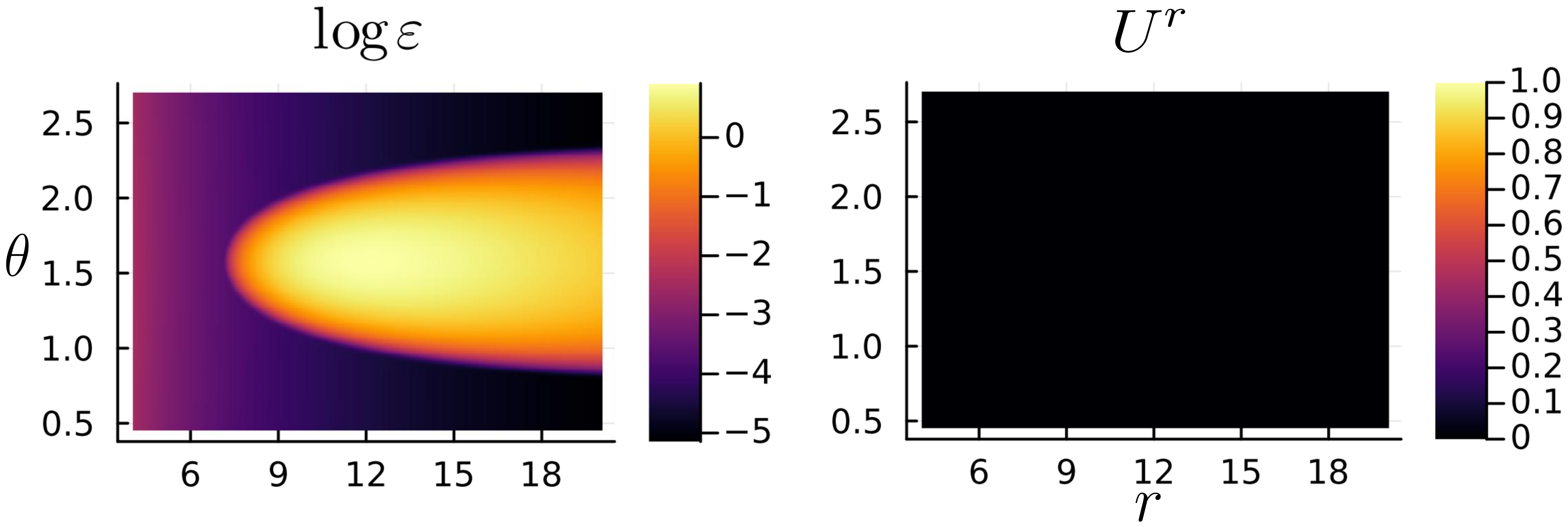
Validation V : Curved Spacetime
Kerr Fat Disks

- Fishborne-Moncrief stationary disk:
  - Known Ideal Hydro stationary solution for purely azimuthal flow in Kerr. $u_\mu = u_t\,(1, 0, 0, -l)$
- Used in testing ideal relativistic hydrodynamics codes
- Our BDNK implementation
  - Initialized with the Ideal solution
  - Near-Extremal a/M=0.99
  - Challenges: Boundary effects, sharp discontinuities, atmosphere
  - Critically, the ideal solution is not a stationary solution in BDNK!



Validation V : Curved Spacetime – Kerr Disks

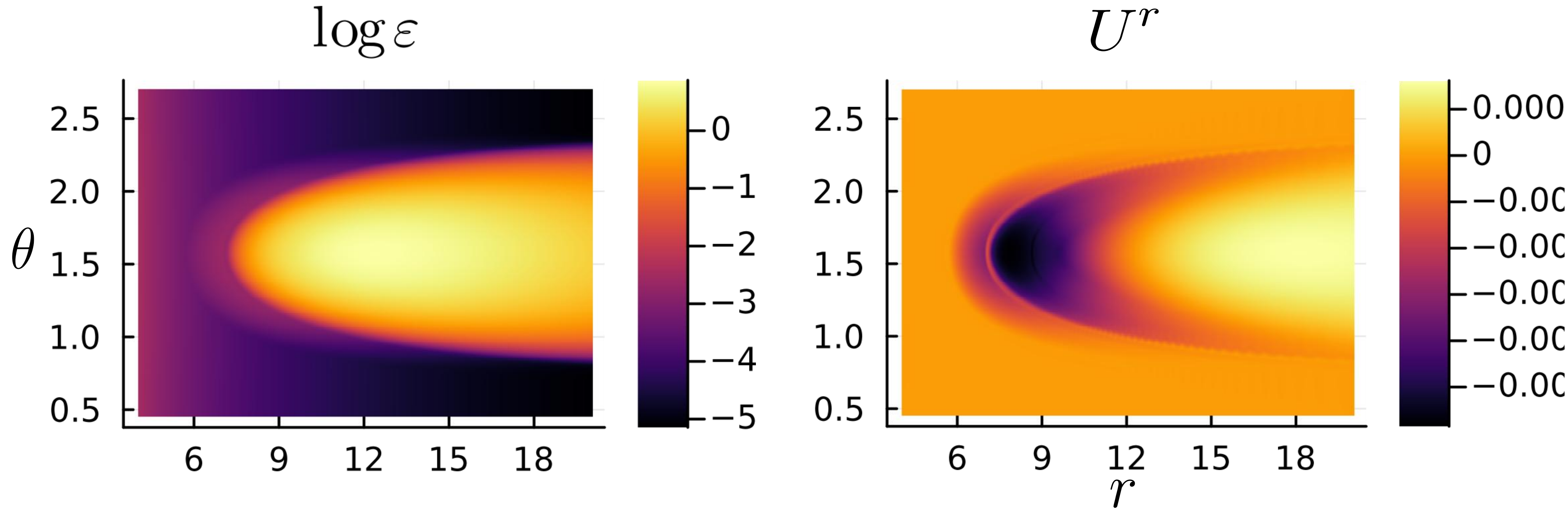
t=0.0 M



- Initial conditions: Ideal stationary Fishborne-Moncrief disk
- In BDNK, small but finite viscosity leads to infall

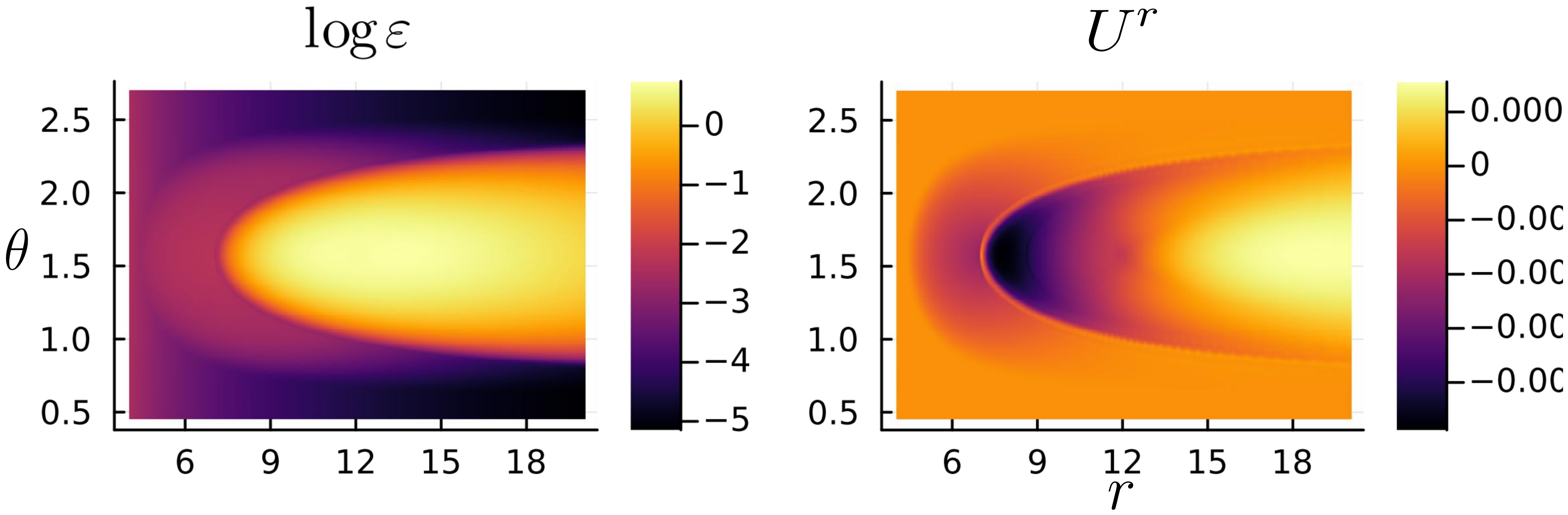
Validation V : Curved Spacetime – Kerr Disks

t=2.5 M



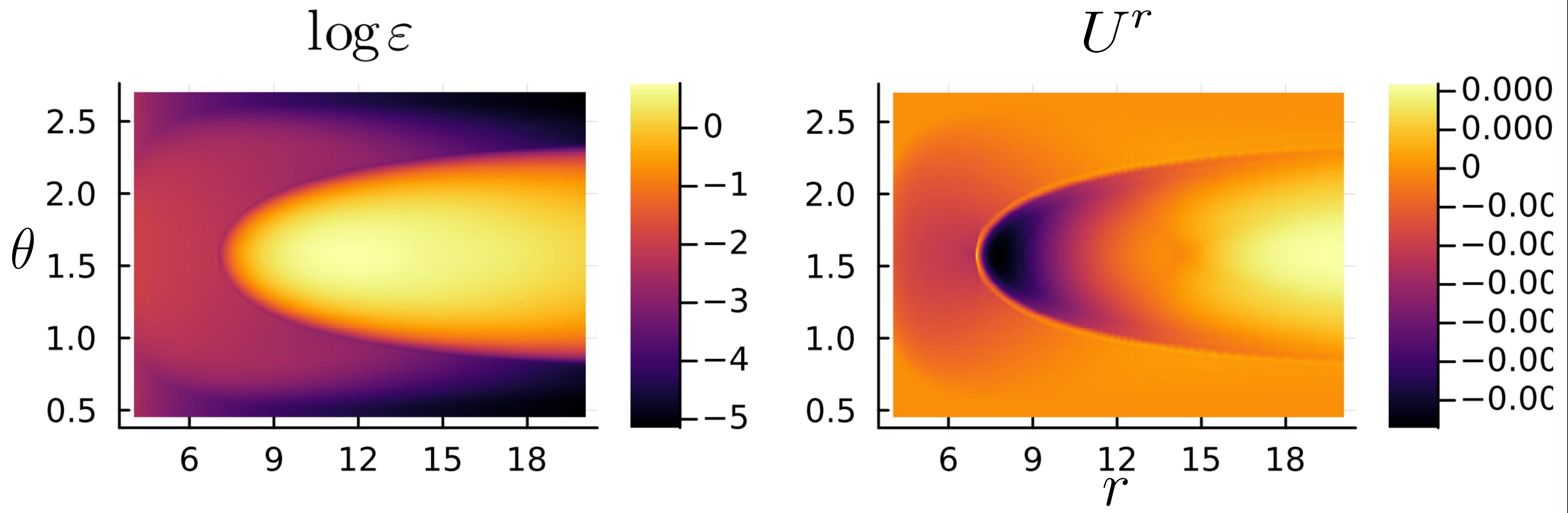
Validation V : Curved Spacetime – Kerr Disks

t=5.0 M

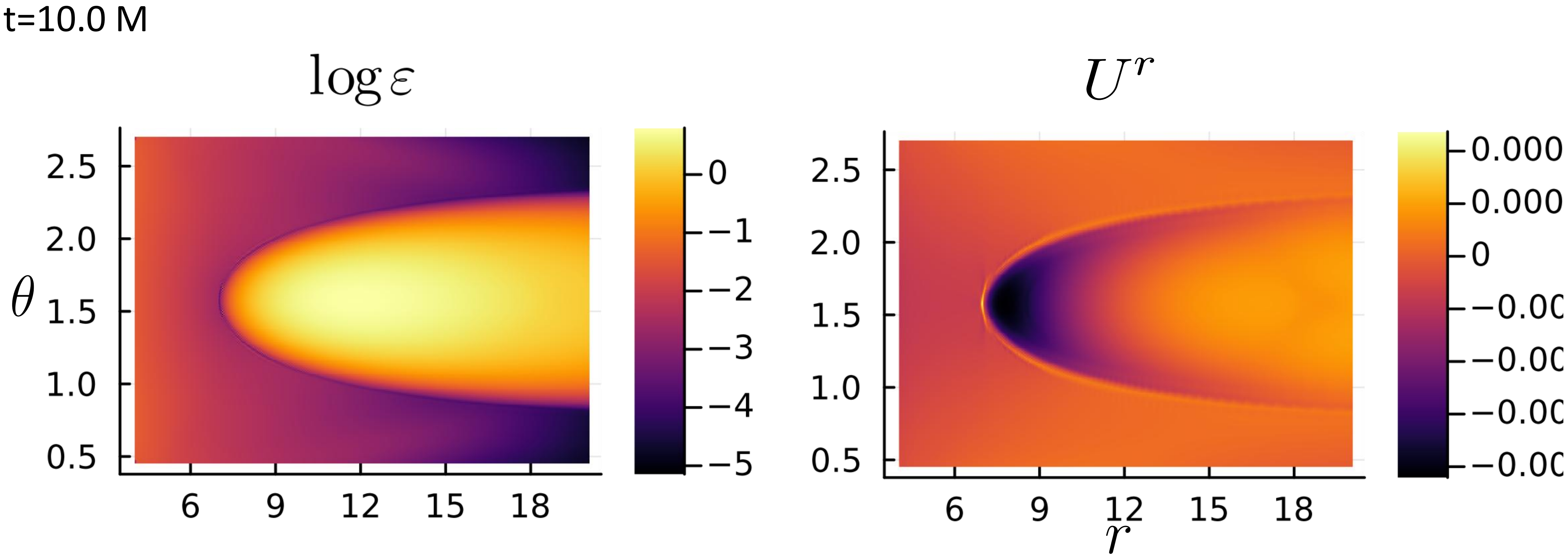


Validation V : Curved Spacetime – Kerr Disks

t=7.5 M

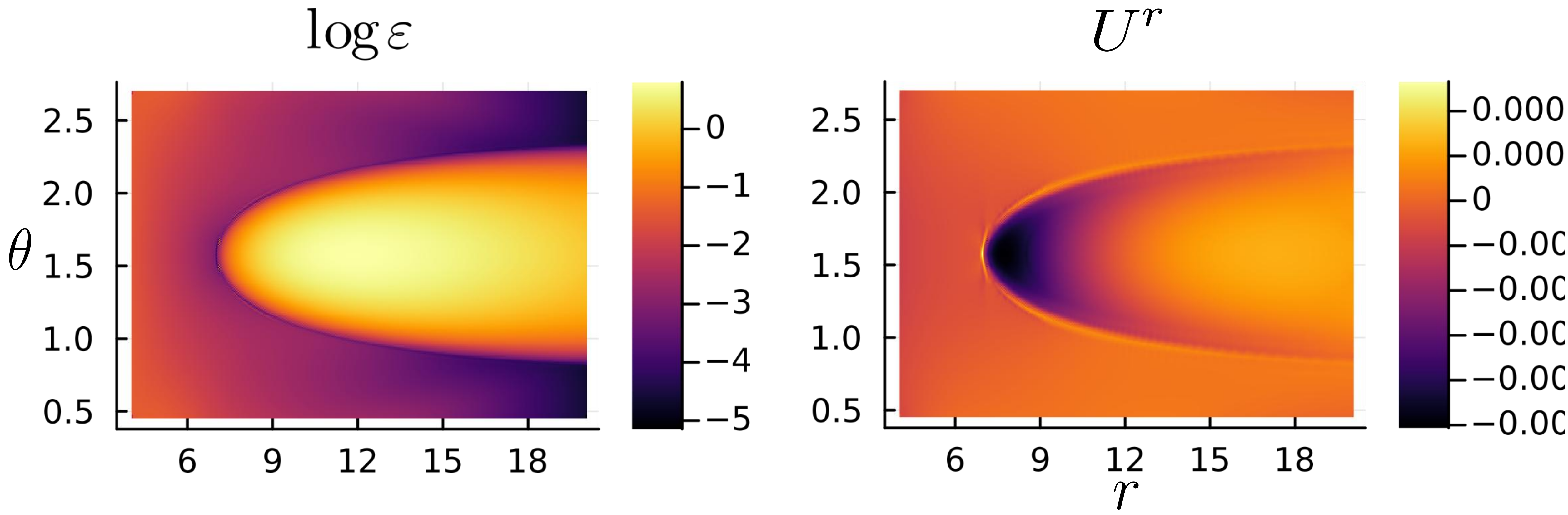


Validation V : Curved Spacetime – Kerr Disks



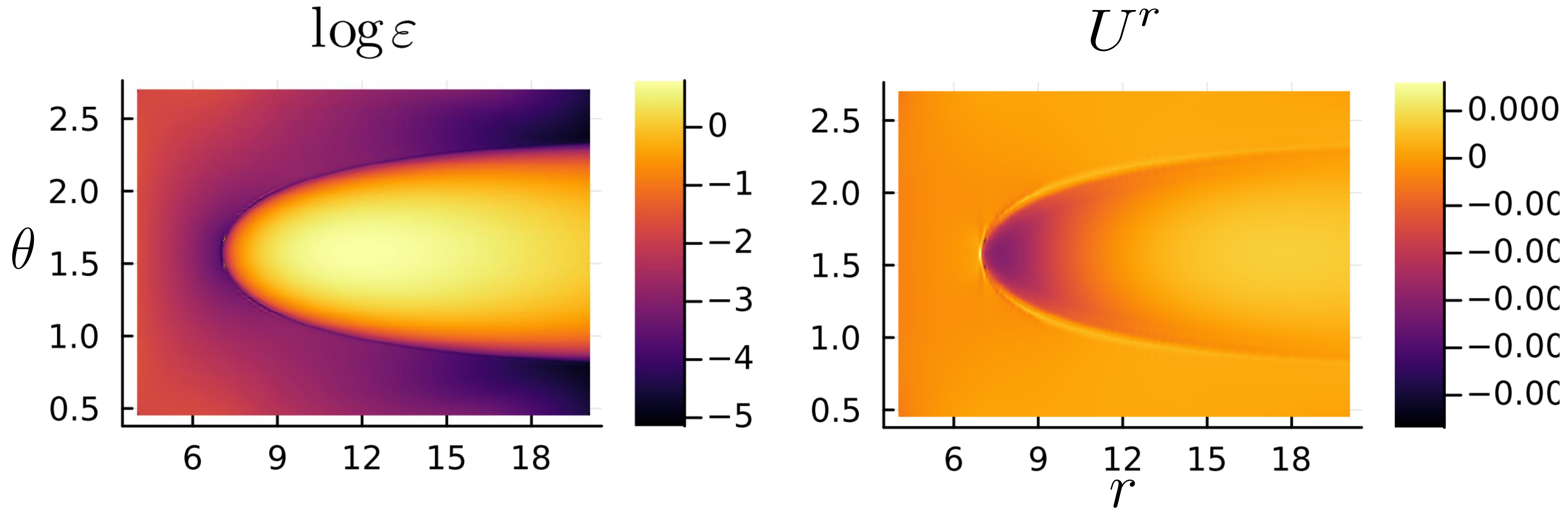
Validation V : Curved Spacetime – Kerr Disks

t=12.5 M



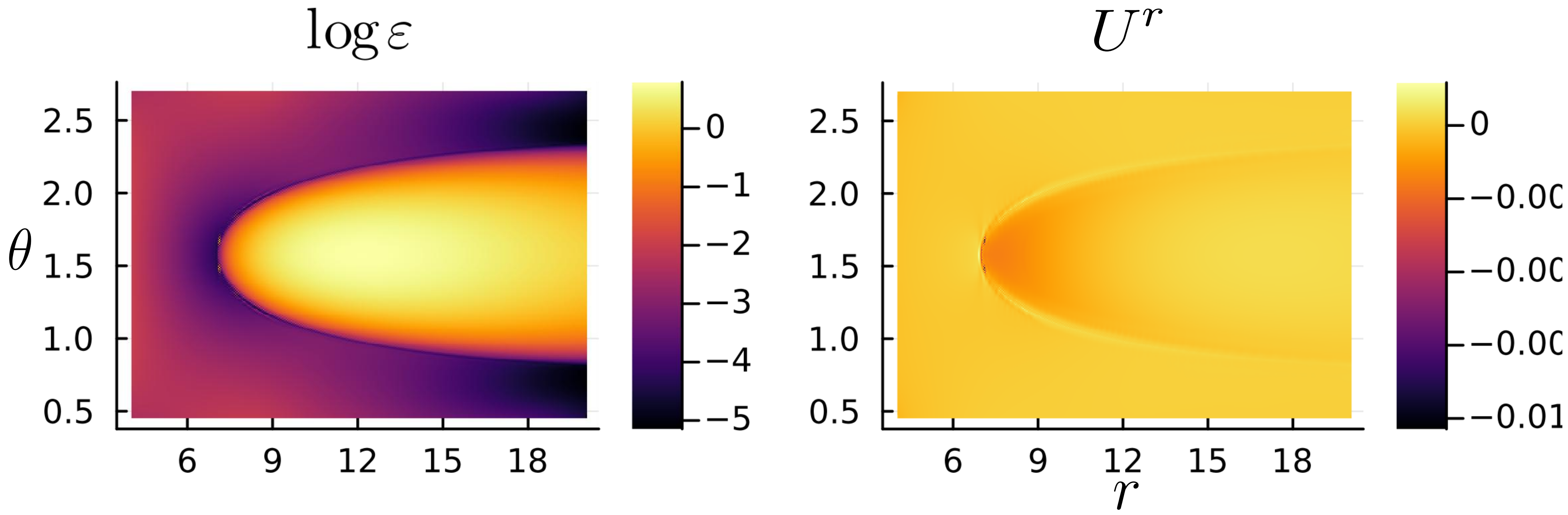
Validation V : Curved Spacetime – Kerr Disks

t=15.0 M



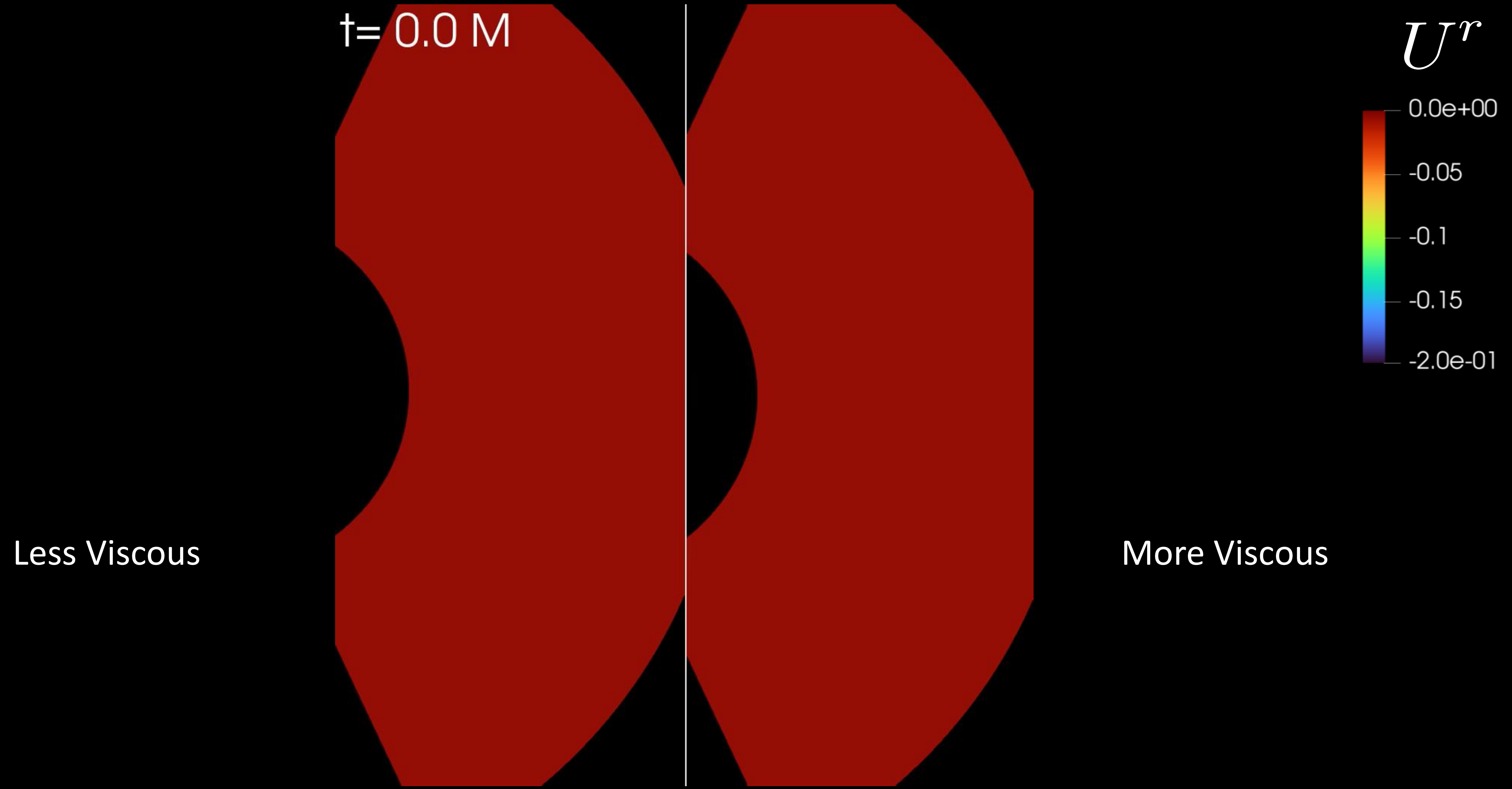
Validation V : Curved Spacetime – Kerr Disks

t=20.0 M



Validation V : Curved Spacetime – Kerr Disks

Comparing viscous effects



Summary

- BDNK : Causal and Stable Relativistic Viscous Hydrodynamics
- 3+1D, general spacetimes, arbitrary EOS
- Validation
  - Reproduced known semi-analytical solutions
  - Low-Viscosity agreement with Ideal-Hydro solutions
  - Comparisons with Israel-Stewart theory for QCD EOS
  - Shock Tube tests

arXiv:26??.xxxx

Outlook

- Open Source Code (HIC and Astrophysics community)
- Applications to Heavy Ion Collision data
- AMR for Large Scale GR-Hydro Simulations
- Visco-resistive MHD formulism
- 2+1D, 3+1D Shock Analysis

Lots of 'exciting' technical challenges remain on the mathematical formulation as well practical implementation and applications!

Summarizing the viscous landscape

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$

IDEAL $T_0^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$

$$\boxed{\dot{\vec{u}} + (\vec{u}\cdot\nabla)\vec{u} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\vec{u}}$$

Relativistic Navier Stokes

$$T_{\mu\nu} = (\varepsilon + \mathcal{A})\, u_\mu u_\nu + (P + \Pi)\Delta_{\mu\nu} - 2\eta\sigma_{\mu\nu} + u_\mu \mathcal{Q}_\nu + u_\nu \mathcal{Q}_\mu$$

	CAUSAL	STABLE	HYPERBOLIC	CODES
Landau/Eckart: Bad Frame Choice $\mathcal{A} = 0$ OR $\mathcal{Q}^\mu = 0$	✗	✗	✗	✓
BDNK: General choice $\mathcal{A} = \tau_\varepsilon \left[u^\mu \nabla_\mu \varepsilon + (\varepsilon + P)\nabla_\mu u^\mu\right]$ Explicit algebraic constraints on transport coefficients	✓	✓	✓	**THIS WORK**
Israel-Stewart theories $\tau_\Pi u^\mu \partial_\mu \Pi + \Pi = -\zeta \partial_\mu u^\mu$				✓

- 10 New degrees of freedom $\Pi^{\mu\nu}$
- Causality conditions only partially known
- Exceedingly complicated GR implementation

Backups

Simulating BDNK

Previous Works

- Pandya et al. '21, '22' — 2+1D : Flat Spacetime
- Bea et al. 2023 — 1+1D,2+1D : QGP
- Keeble et al. 2025, Shumet et al. 2025. — 1+1D : Neutron star

Our Work : General 3+1 Dimensional Relativistic Simulations

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$
$$\nabla_\mu T^{\mu\nu} = 0$$
8 DOFS

Finite Volume Method (FVM)

- Extensively used in relativistic and non-relativistic simulations.
- Equations cast in a first-order flux-conservative form.
- Complicated in BDNK : Second order PDE system.
- Our FVM: KT scheme used in MUSIC

$$\underbrace{\partial_t \mathbf{Q}}_{\text{Variable}} + \underbrace{\partial_i \mathbf{F_i}}_{\text{Flux}} = \underbrace{\mathbf{S}}_{\text{Sources}}$$

$Y^1 \equiv \partial_t U^1$ **FIRST ORDER** $X_i^1 \equiv \partial_i U^1$

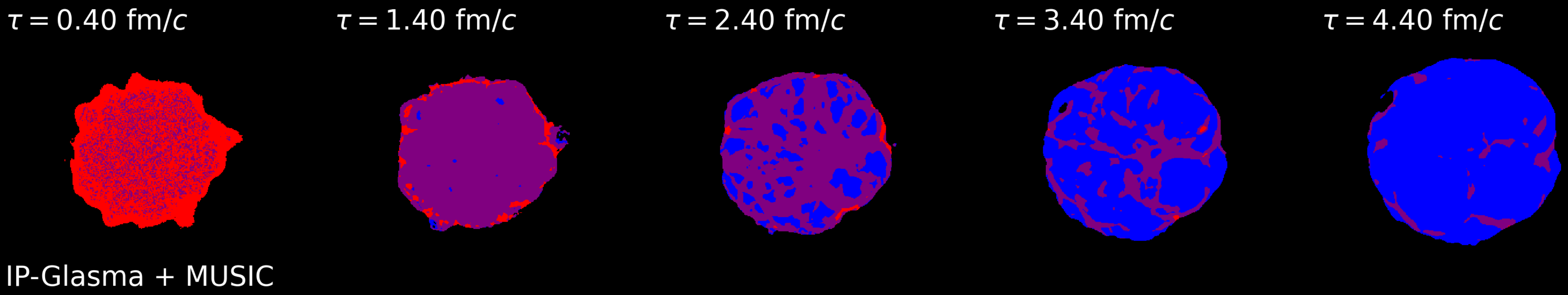
- Spatial gradients are treated as independent variables
- New variables lead to new constraint equations
- Need additional auxiliary variables and extra EOMs

44 Variables $\{T^{0\mu}, X_i^A, \log\epsilon, U^i, n, \varphi_i^A, \phi^A, \Psi_i, \psi\}$

$Y^1 \equiv \partial_t U^1$ **HYBRID / MIXED**

- Spatial derivatives are numerically evaluated
- Not exactly flux-conservative : Sources have derivatives
- Fewer variables : Computationally cheaper

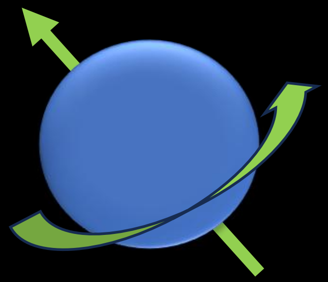
9 Variables $\{T^{0\mu}, \log\epsilon, U^i, n\}$



Plumber et al, 2021

Causality violations in realistic simulations of heavy-ion collisions

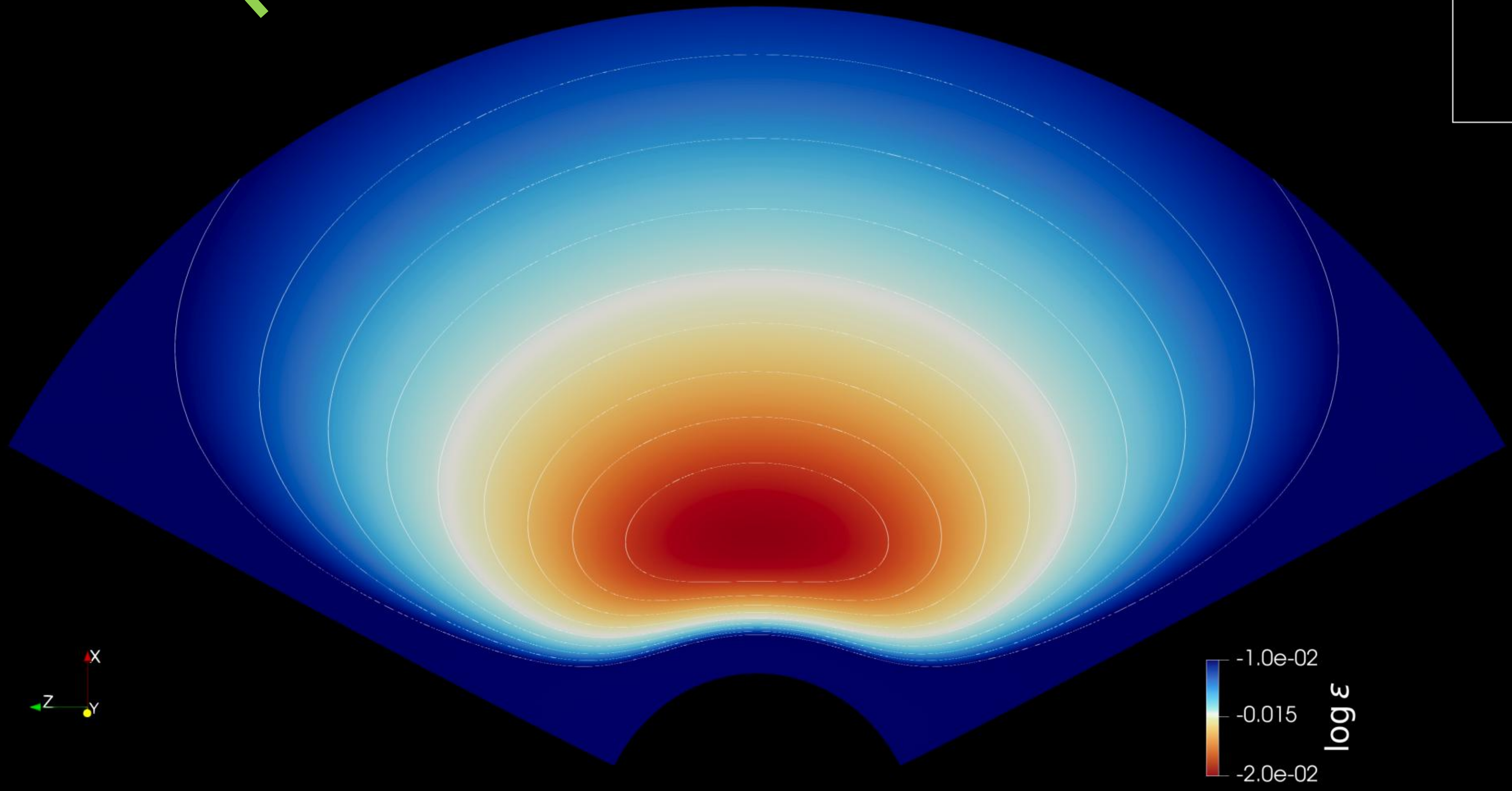
Validation V : Curved Spacetime – Kerr Disks



In **Boyer–Lindquist coordinates** (t, r, θ, ϕ), the Kerr line element is

$$\boxed{\begin{aligned} ds^2 = & -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}\, dt\, d\phi \\ & + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 \\ & + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2, \end{aligned}}$$

$$r_\pm = M \pm \sqrt{M^2 - a^2} \qquad \frac{r_+}{M} \approx 1.141$$



- Ideal Fishborne-Moncrief stationary disk
- Near-Extremal a/M=0.99
- Used in testing ideal relativistic hydrodynamics codes
- BDNK implementation challenges: Boundary effects, sharp discontinuities, astmosphere

Definition of out-of-equilibrium variables: Hydrodynamic Frame

$$T_0^{\mu\nu} = (\varepsilon_{eq} + P_{eq}) u^\mu_{eq} u^\nu_{eq} + P_{eq} g^{\mu\nu}$$

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$

- Fictitious $T,\, u^\mu$
- Ambiguity in parameterizing fluid stress-energy tensor, provided one recovers the unambiguous equilibrium ideal fluid tensor in equilibrium.
- Different choices correspond to different *effective* descriptions.

Minimal IS theory : MUSIC

$$T_{\mathrm{IS}}^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$\tau_\Pi u^\mu \partial_\mu \Pi + \Pi = -\zeta \partial_\mu u^\mu$$

$$\tau_\pi u^\sigma \partial_\sigma \left(\Delta^{\mu\nu}_{\alpha\beta} \pi^{\alpha\beta}\right) + \pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$$

BDNK

$$T_{\mu\nu} = (\varepsilon + \mathcal{A})\, u_\mu u_\nu + (P + \Pi)\Delta_{\mu\nu} - 2\eta\sigma_{\mu\nu} +$$

$$\mathcal{A} = \tau_\varepsilon \left[u^\mu \nabla_\mu \varepsilon + (\varepsilon + P)\nabla_\mu u^\mu\right] = -\tau_\varepsilon \left(u_\alpha \nabla_\beta T_0^{\alpha\beta}\right)$$

Comparison with IS

- Advantages of BDNK vs IS (Implementation)
  - Simpler GR versions of the equations
  - Simpler non-linear causality and stability conditions
- Equal initial conditions (Recovery)
  - Straightforward in IS
  - Tricky in BDNK
  - Approach here:

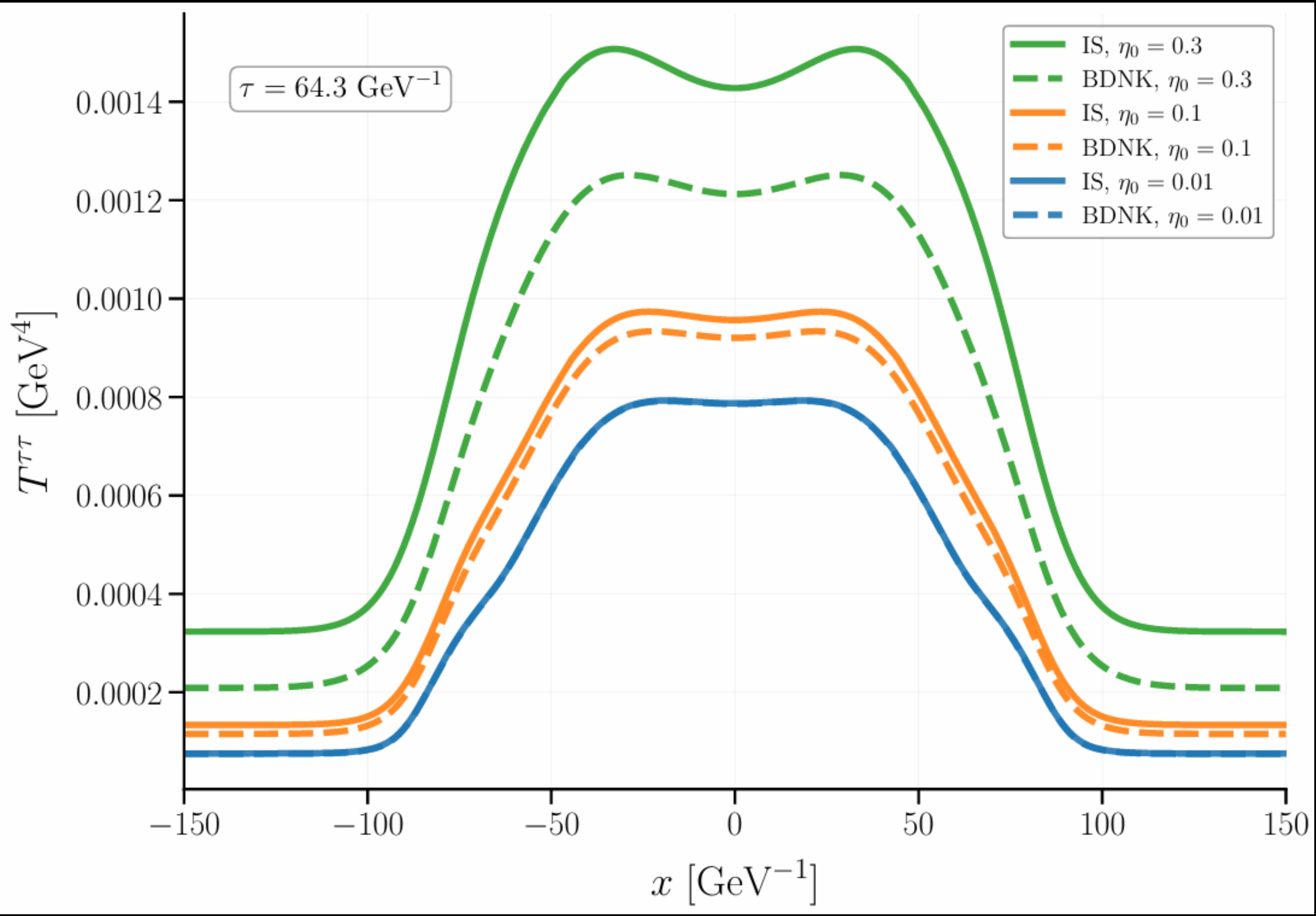
$$\{\varepsilon(\tau_0), u^\mu(\tau_0); \tau_{BDNK}\}$$

$$T^{\mu\nu}_{BDNK}(\tau_0) = T^{\mu\nu}_{IS}(\tau_0)$$

$$\varepsilon(\tau_0), u^\mu(\tau_0), \Pi, \pi^{\mu\nu}; \tau_{IS}\}$$

Minimal IS theory : MUSIC

Gaussians: BDNK vs IS (Hot QCD EOS)



$$T^{\mu\nu}_{\rm IS} = \varepsilon u^\mu u^\nu + (P+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$\tau_\Pi u^\mu \partial_\mu \Pi + \Pi = -\zeta \partial_\mu u^\mu$$

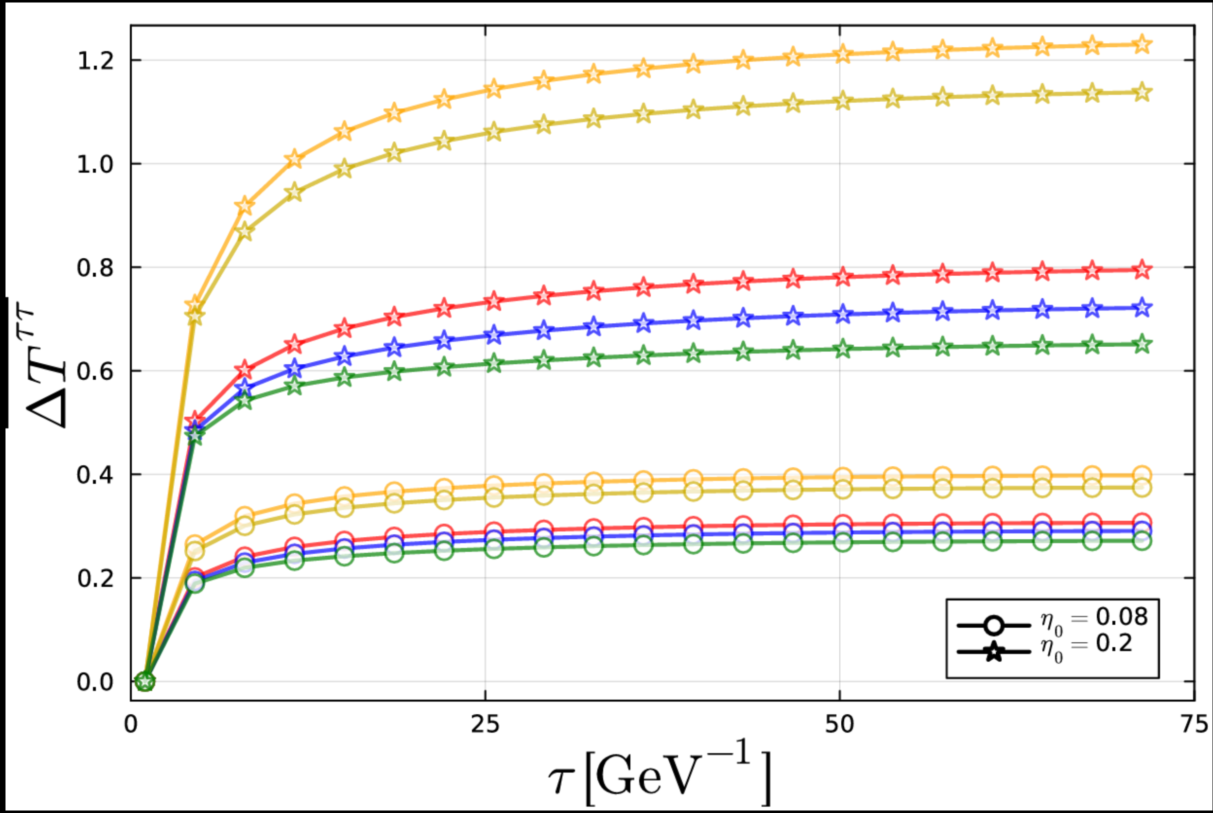
$$\tau_\pi u^\sigma \partial_\sigma \left(\Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha\beta}\right) + \pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$$

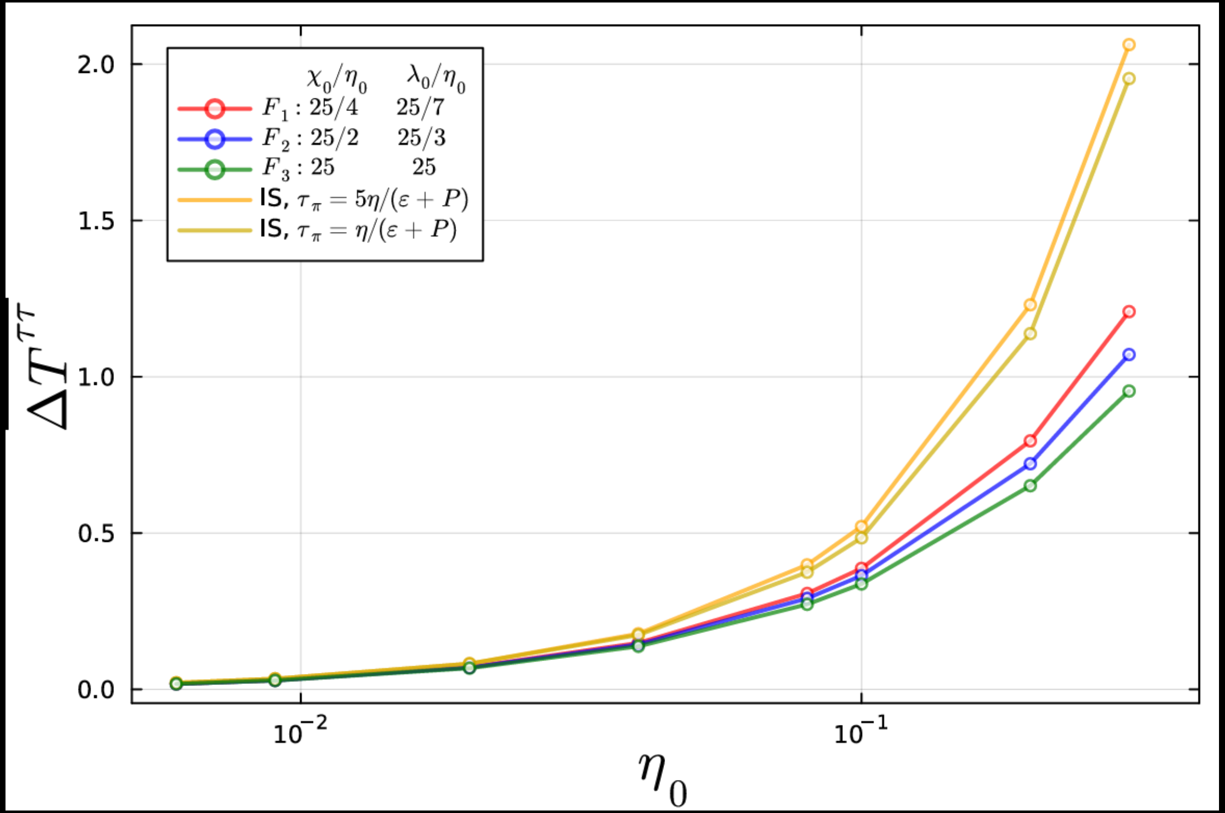
Comparison with IS

- Largest differences coming from initial expansion
- BDNK approaches equilibrium faster
  - Because of finite time relaxation in IS

$$\overline{\Delta u}(t_n) := \frac{\sum_i^N |U_i^n - U_{0,i}^n|}{\sum_i^N |U_{0,i}^n|}$$

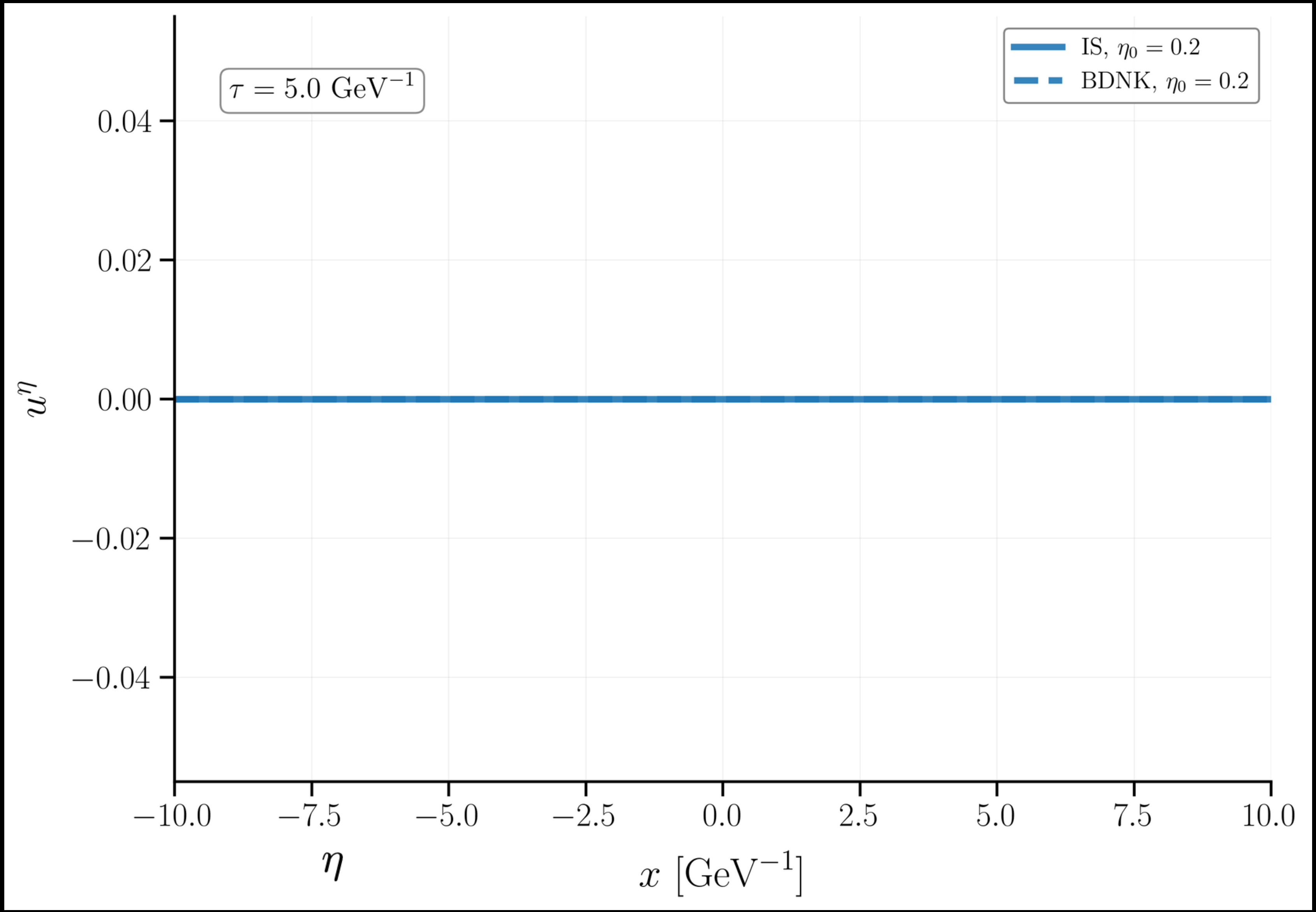
Normalized absolute difference w.r.t ideal averaged over the domain



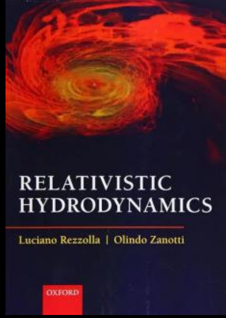


Comparison with IS

- Comparison in the longitudinal direction: Velocity
- MUSIC: Artifacts in near the edges



Causality & Stability Illustration



$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$

IDEAL

Viscous Terms

Eckart heat-flux constitutive relation, schematically

$$a^\mu = u^\nu \nabla_\nu u^\mu$$

$$u_\mu q^\mu = 0$$

$$\boxed{q^\mu = -\kappa \left(\Delta^{\mu\nu}\nabla_\nu T + T a^\mu\right)}$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}.$$

LRF $\quad \mathbf{q} = -\kappa \nabla T. \qquad u^\mu = (1, \mathbf{0}).$

$$T^{00} = \epsilon, \qquad T^{i0} = q^i.$$

$$c_V \partial_t T - \kappa \nabla^2 T = 0. \quad \leftarrow \quad \boxed{\partial_t \epsilon + \nabla \cdot \mathbf{q} = 0.} \quad \leftarrow \quad \partial_t T^{00} + \partial_i T^{i0} = 0,$$

Heat Diffusion Equation (Parabolic)

$$c_V \equiv \left(\frac{\partial \epsilon}{\partial T}\right)_V,$$

$$G(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right), \qquad t > 0.$$

It corresponds to the initial condition

$$G(x,0) = \delta(x).$$

Instantaneous response everywhere!

$$\boxed{T(x,t) = \int_{-\infty}^{\infty} G(x - x', t)\, T_0(x')\, dx'}$$

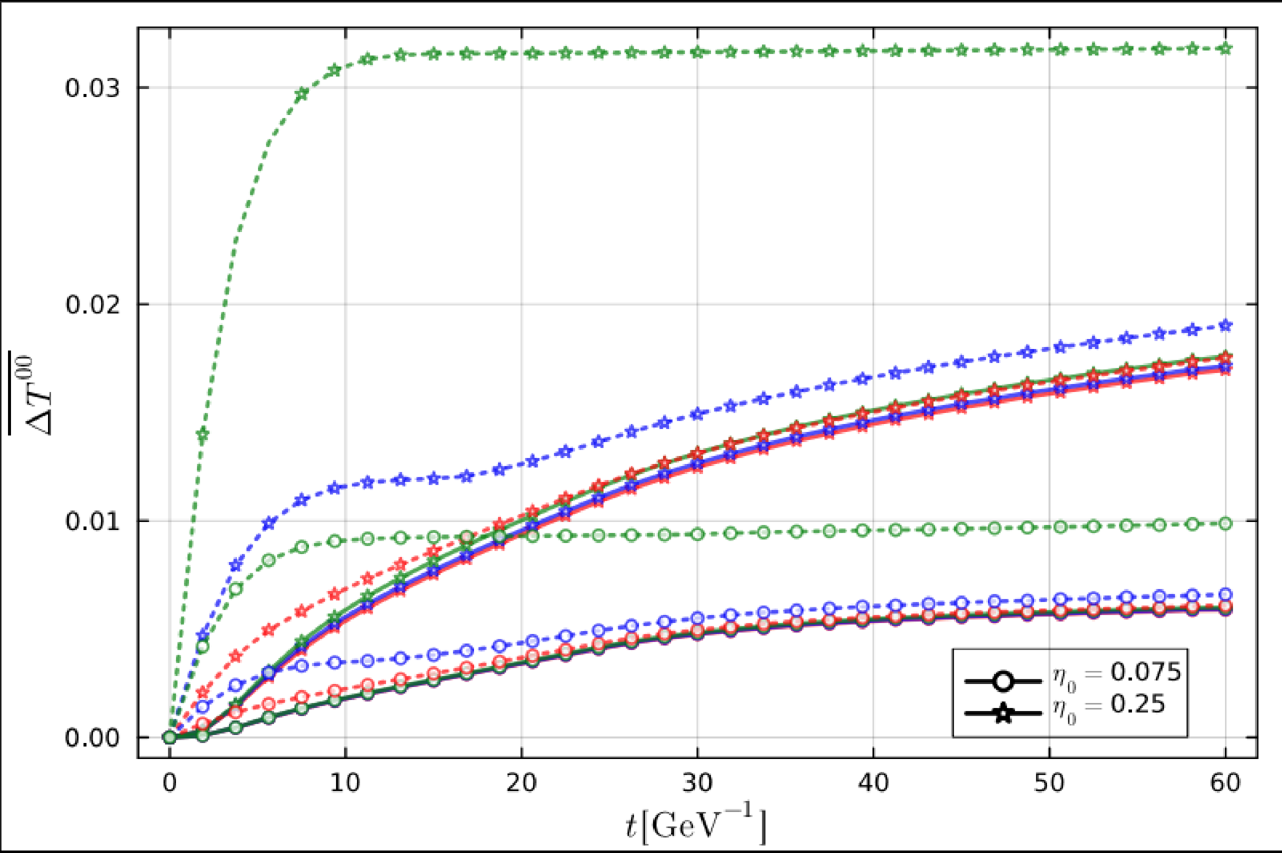
$$\boxed{T(t,x) \neq 0 \quad \forall x,\ t > 0}$$

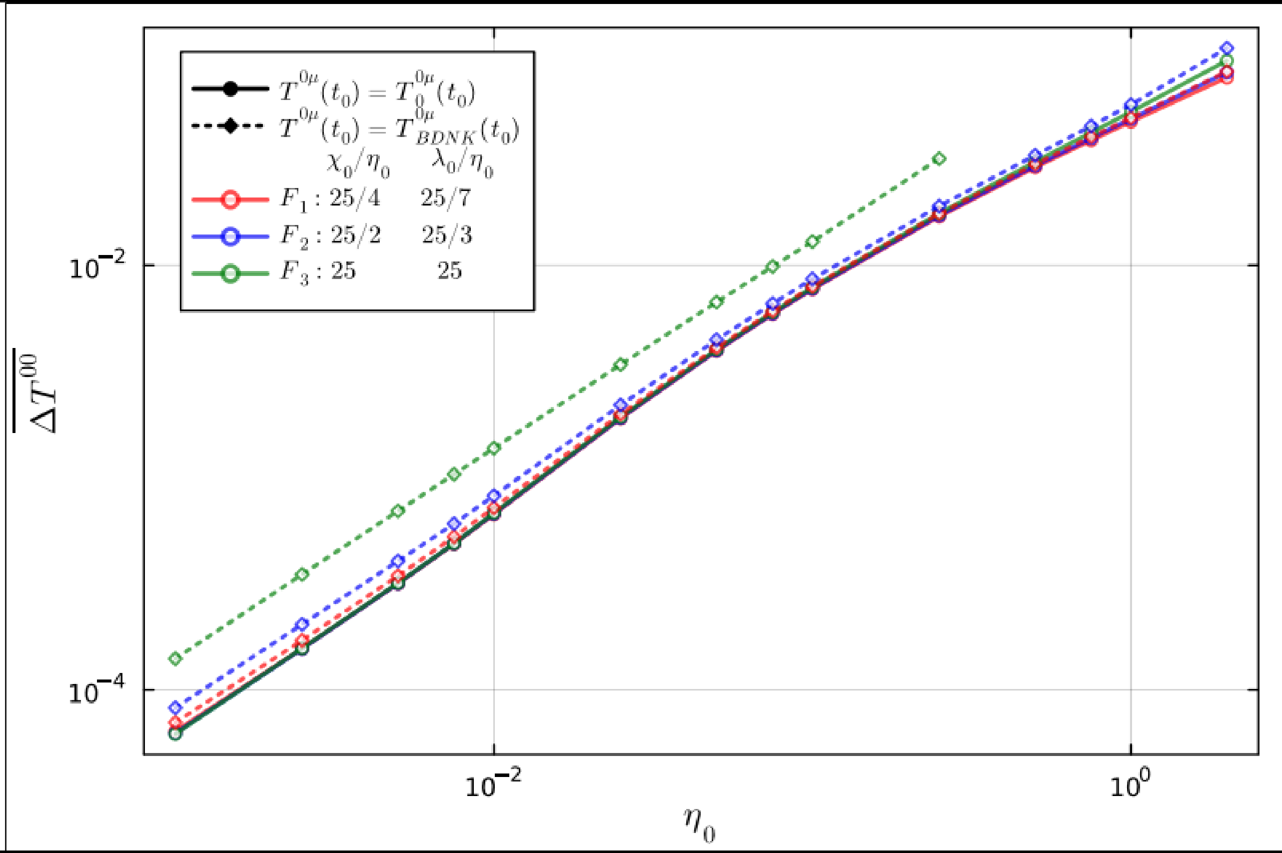
Validation III: Shock Tube tests

Quantifying departure from Ideal Hydrodynamics

$$\Delta u(t_n) := \frac{\langle |U_i^n - U_{0,i}^n| \rangle}{\langle |U_{0,i}^n| \rangle}$$

Relative absolute difference averaged over the grid





χ_0 λ_0

BDNK Frame

(Causality and Stability)

Independence of frame parameters : Frame Robust

✓ $T^{0\mu}(t_0, \vec{\mathbf{x}}) = T_0^{0\mu}(t_0, \vec{\mathbf{x}})$

$T^{0\mu}(t_0, \vec{\mathbf{x}}; \eta_0, \chi_0, \lambda_0)$ ✗

Viscous Transport Coefficients

η_0 $\zeta = \sigma = 0$

Ideal/Inviscid $\eta_0 \to 0$

Kelvin Helmholtz Setup

Adopted from Lecoanet et. al. 2015

$$\epsilon = 1$$

$$U^x = u_{\text{flow}}\left[\tanh\left(\frac{y-y_1}{a}\right) - \tanh\left(\frac{y-y_2}{a}\right) - 1\right]$$

$$U^y = A\sin(2\pi x)\left[\exp\left(-\left[\frac{y-y_1}{\sigma}\right]^2\right) + \exp\left(-\left[\frac{y-y_2}{\sigma}\right]^2\right)\right],$$

BDNK EOMs – Constitutive Relations

$$T_{\mu\nu} = (\varepsilon + \mathcal{A})\, u_\mu u_\nu + (P + \Pi)\Delta_{\mu\nu} - 2\eta\sigma_{\mu\nu} + u_\mu \mathcal{Q}_\nu + u_\nu \mathcal{Q}_\mu$$

$$\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

$$\Delta^{\mu\nu\alpha\beta} = \left(\Delta^{\alpha\mu}\Delta^{\beta\nu} + \Delta^{\alpha\nu}\Delta^{\beta\mu}\right)/2 - \Delta^{\mu\nu}\Delta^{\alpha\beta}/3$$

$$\sigma^{\mu\nu} = \Delta^{\mu\nu\alpha\beta}\nabla_\alpha u_\beta$$

$$\mathcal{A} = \tau_\varepsilon \left[u^\mu \nabla_\mu \varepsilon + (\varepsilon + P)\nabla_\mu u^\mu\right] = -\tau_\varepsilon \left(u_\alpha \nabla_\beta T_0^{\alpha\beta}\right)$$

$$\Pi = \frac{\tau_P}{\tau_\varepsilon}\mathcal{A} - \zeta\nabla_\mu u^\mu = -\zeta\nabla_\mu u^\mu - \tau_P \left(u_\alpha \nabla_\beta T_0^{\alpha\beta}\right)$$

$$\mathcal{Q}_\nu = \tau_Q(\varepsilon + P)u^\lambda \nabla_\lambda u_\nu + \beta_\varepsilon \Delta_\nu^\lambda \nabla_\lambda \varepsilon + \beta_n \Delta_\nu^\lambda \nabla_\lambda n$$

Transport coefficients – Simplifying choices

$$\tau_\epsilon = \chi V/\rho\,, \quad \tau_Q = \lambda V/\rho\,, \quad \tau_P = \xi\tau_\epsilon$$

$$\lambda > \frac{\chi}{\chi(1-c_s^2)(1-\xi) - P'_\epsilon}$$

$$\chi > P'_\varepsilon \mathrm{max}\left[\frac{1}{(1-c_s^2)(1-\xi)}\,,\ \frac{1}{c_s^2\xi}\right]$$

$$\sigma = 0$$

$$\xi = P'_\varepsilon = c_s^2 = \frac{1}{3}\,, \qquad \chi = \frac{3}{4}\frac{\chi_0}{\eta_0}\,, \qquad \lambda = \frac{3}{4}\frac{\lambda_0}{\eta_0}\,,$$

$$\frac{\chi_0}{\eta_0} \geq 4, \qquad \frac{\lambda_0}{\eta_0} \geq \frac{3\chi_0}{\chi_0 - \eta_0}$$

BDNK EOMs – H Tensor

$$T_{\mu\nu} = (\varepsilon + \mathcal{A})\, u_\mu u_\nu + (P + \Pi)\Delta_{\mu\nu} - 2\eta\sigma_{\mu\nu} + u_\mu \mathcal{Q}_\nu + u_\nu \mathcal{Q}_\mu$$

$$T^{\mu\nu} = T_0^{\mu\nu} + \frac{1}{\varepsilon} H^{\mu\nu\alpha\beta}\nabla_\alpha(\varepsilon u_\beta) + 2\beta_n u^{(\mu}\Omega^{j\nu)}\partial_j n$$

$$\begin{aligned} H^{\mu\nu\alpha\beta} &= \tau_\varepsilon u^\mu u^\nu \left[(\varepsilon + P)\Delta^{\alpha\beta} - \varepsilon u^\alpha u^\beta\right] + \Delta^{\mu\nu}\left[(\tau_P(\varepsilon + P) - \zeta)\Delta^{\alpha\beta} - \varepsilon\tau_P u^\alpha u^\beta\right] - 2\eta\Delta^{\mu\nu\alpha\beta} \\ &+ \tau_Q(\varepsilon + P)u^\mu u^\alpha \Delta^{\beta\nu} + \tau_Q(\varepsilon + P)u^\nu u^\alpha \Delta^{\mu\beta} - \varepsilon\beta_\varepsilon u^\mu u^\beta \Delta^{\alpha\nu} - \varepsilon\beta_\varepsilon u^\nu u^\beta \Delta^{\mu\alpha} \\ &- \frac{n\beta_n}{u^0}\left(u^\mu \Delta^{\nu 0} + u^\nu \Delta^{\mu 0}\right)\Delta^{\alpha\beta} \end{aligned}$$

$$C_i^\mu = \delta_i^\mu - \frac{\delta_0^\mu u_i}{u_0} = \Delta_i^\mu - \frac{\Delta_0^\mu u_i}{u_0}, \quad \left(C_{i\mu} = g_{\alpha\mu}C_i^\alpha = g_{\mu i} - \frac{g_{\mu 0}u_i}{u_0}\right).$$

$$P_{A\mu} = \begin{cases} u_\mu \text{ if } A = \theta \\ C_{i\mu} \text{ if } A = i \end{cases} \quad \text{and} \quad \tilde{P}^{A\mu} = \begin{cases} -u^\mu \text{ if } A = \theta \\ \Delta^{i\mu} \text{ if } A = i \end{cases}.$$

Divergence Cleaning

$$\partial_t \mathbf{B} - \nabla \times \mathbf{E} = 0 \qquad\qquad \nabla \cdot \mathbf{B} = 0$$

Dedner et al. (2002) : GLM methods to resolve numerical errors causing instability in MHD simulations

$$\partial_t \mathbf{B} - \nabla \times \mathbf{E} + \nabla \phi = 0,$$
$$\partial_t \phi + c_h^2 \nabla \cdot \mathbf{B} = -\frac{\phi}{\tau},$$

- Full first-order formulation introduces **curl constraints**
- Introduce an auxiliary **curl-cleaning (CC) field** to propagate and damp constraint violations
- Cleaning/damping controlled by adjustable coefficients

$c_h \qquad \tau$

$\nabla \cdot \mathbf{B} \approx 0$

First Order Formulism – Curl Cleaning

$$\mathbf{X} = \nabla\psi \qquad\qquad \nabla \times \mathbf{X} = 0$$

New variables which are gradients requires methods to ensure curl is zero

$$\begin{aligned} \partial_t Y - \nabla \cdot \mathbf{X} &= -\psi \\ \partial_t \mathbf{X} - \nabla Y &= 0 \\ \partial_t \psi &= Y. \end{aligned}$$

$$\begin{aligned} Y &= \partial_t \psi, \\ \mathbf{X} &= \nabla \psi, \end{aligned}$$

Ensure Curl does not grow

$$\begin{aligned} \partial_t \mathbf{X} - \nabla Y + \nabla \times \varphi &= 0, \\ \partial_t \varphi - c_\varphi^2 \nabla \times \mathbf{X} &= -\frac{\varphi}{\tau_1}, \end{aligned}$$

$$\begin{aligned} \partial_t^2 (\nabla \times \varphi) - c_\varphi^2 \nabla^2 (\nabla \times \varphi) &= -\frac{\partial_t (\nabla \times \varphi)}{\tau_1}, \\ \partial_t^2 (\nabla \cdot \varphi) &= -\frac{\partial_t (\nabla \cdot \varphi)}{\tau_1}. \end{aligned}$$

First Order Formulism – Curl Cleaning

$$Y^A \equiv \partial_t U^A,$$
$$X_i^A \equiv \partial_i U^A,$$
$$Z_i \equiv \partial_i n.$$

New Variables for flux conservative form

BDNK Tensor

$$T^{\mu\nu} = T_0^{\mu\nu} + H^{\mu\nu 0\beta} P_{A\beta} Y^A + H^{\mu\nu j\beta} P_{A\beta} X_j^A + 2\beta_n u^{(\mu}\Omega^{j\nu)} Z_j + H^{\mu\nu\alpha\beta} C_{j\beta} \Gamma^j_{\alpha\lambda} u^\lambda$$

Auxiliary Definitions

$$Y^A = \tilde{P}^{A\lambda} M^{-1}_{\lambda\mu} \left[T^{0\mu} - T_0^{0\mu} - H^{\mu 0 k\beta} P_{B\beta} X_k^B - 2\beta_n u^{(\mu}\Omega^{k0)} Z_k - H^{\mu 0\alpha\beta} C_{k\beta} \Gamma^k_{\alpha\lambda} u^\lambda \right],$$
$$T^{ij} = T_0^{ij} + H^{ij0\beta} P_{A\beta} Y^A + H^{ijk\beta} P_{A\beta} X_k^A + 2\beta_n u^{(i}\Omega^{kj)} Z_k + H^{ij\alpha\beta} C_{k\beta} \Gamma^k_{\alpha\lambda} u^\lambda.$$

EOMs

$$\partial_t T^{0\mu} + \partial_i T^{i\mu} = -\Gamma^\nu_{\nu 0} T^{0\mu} - \Gamma^\mu_{0\nu} T^{0\nu} - \Gamma^\nu_{\nu i} T^{i\mu} - \Gamma^\mu_{i\nu} T^{i\nu},$$
$$\partial_t X_i^A - \partial_i Y^A = 0,$$
$$\partial_t Z_i - \partial_i S_n = 0,$$
$$\partial_t U^A = Y^A,$$
$$\partial_t n = S_n.$$

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$$Y^A \equiv \partial_t U^A,$$
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Auxiliary Definitions

$$Y^A = \tilde{P}^{A\lambda} M^{-1}_{\lambda\mu} \left[T^{0\mu} - T_0^{0\mu} - H^{\mu 0 k\beta} P_{B\beta} X_k^B - 2\beta_n u^{(\mu}\Omega^{k0)} Z_k - H^{\mu 0\alpha\beta} C_{k\beta} \Gamma^k_{\alpha\lambda} u^\lambda \right],$$
$$T^{ij} = T_0^{ij} + H^{ij0\beta} P_{A\beta} Y^A + H^{ijk\beta} P_{A\beta} X_k^A + 2\beta_n u^{(i}\Omega^{kj)} Z_k + H^{ij\alpha\beta} C_{k\beta} \Gamma^k_{\alpha\lambda} u^\lambda.$$

EOMs

$$\partial_t T^{0\mu} + \partial_i T^{i\mu} = -\Gamma^\nu_{\nu 0} T^{0\mu} - \Gamma^\mu_{0\nu} T^{0\nu} - \Gamma^\nu_{\nu i} T^{i\mu} - \Gamma^\mu_{i\nu} T^{i\nu},$$
$$\partial_t X_i^A - \partial_i Y^A = 0,$$
$$\partial_t Z_i - \partial_i S_n = 0,$$
$$\partial_t U^A = Y^A,$$
$$\partial_t n = S_n.$$

Flux Conservative Form

44 Variables

$$\partial_t \mathcal{Q} + \partial_i F_i = \mathcal{S},$$

$$\mathcal{Q} = \begin{bmatrix} T^{0\mu} \\ X_1^A \\ X_2^A \\ X_3^A \\ Z_1 \\ Z_2 \\ Z_3 \\ U^A \\ n \\ \varphi_1^A \\ \varphi_2^A \\ \varphi_3^A \\ \phi^A \\ \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \psi \end{bmatrix}, \quad F_i = \begin{bmatrix} T^{i\mu} \\ \epsilon_{1ik}\varphi_k^A - \delta_{1i}Y^A \\ \epsilon_{2ik}\varphi_k^A - \delta_{2i}Y^A \\ \epsilon_{3ik}\varphi_k^A - \delta_{3i}Y^A \\ \epsilon_{1ik}\Psi_k - \delta_{1i}S_n \\ \epsilon_{2ik}\Psi_k - \delta_{2i}S_n \\ \epsilon_{3ik}\Psi_k - \delta_{3i}S_n \\ 0_{4\times 1} \\ 0 \\ -\kappa_1^A\epsilon_{1ik}X_k^A + \delta_{1i}\phi^A \\ -\kappa_1^A\epsilon_{2ik}X_k^A + \delta_{2i}\phi^A \\ -\kappa_1^A\epsilon_{3ik}X_k^A + \delta_{3i}\phi^A \\ \kappa_2^A\varphi_i^A \\ -\lambda_1\epsilon_{1ik}Z_k + \delta_{1i}\psi \\ -\lambda_1\epsilon_{2ik}Z_k + \delta_{2i}\psi \\ -\lambda_1\epsilon_{3ik}Z_k + \delta_{3i}\psi \\ \lambda_2\Psi_i \end{bmatrix}, \quad \text{and} \quad \mathcal{S} = \begin{bmatrix} S^\mu \\ 0_{4\times 1} \\ 0_{4\times 1} \\ 0_{4\times 1} \\ 0 \\ 0 \\ 0 \\ Y^A \\ S_n \\ -\beta_1^A\varphi_1^A \\ -\beta_1^A\varphi_2^A \\ -\beta_1^A\varphi_3^A \\ -\beta_2^A\phi^A \\ -\sigma_1\Psi_1 \\ -\sigma_1\Psi_2 \\ -\sigma_1\Psi_3 \\ -\sigma_2\psi \end{bmatrix}$$

Symbolic Expressions

BDNK Tensor

$$T^{\mu\nu} = T_0^{\mu\nu} + \frac{1}{\varepsilon} H^{\mu\nu\alpha\beta}\nabla_\alpha(\varepsilon u_\beta) + 2\beta_n u^{(\mu}\Omega^{j\nu)}\partial_j n$$

$$T^{\mu\nu} = T_0^{\mu\nu} + H^{\mu\nu 0\beta} P_{A\beta} Y^A + H^{\mu\nu j\beta} P_{A\beta} X_j^A + 2\beta_n u^{(\mu}\Omega^{j\nu)} Z_j + H^{\mu\nu\alpha\beta} C_{j\beta}\Gamma^j_{\alpha\lambda} u^\lambda$$

New Variables

$$Y^A \equiv \partial_t U^A,$$
$$X_i^A \equiv \partial_i U^A,$$
$$Z_i \equiv \partial_i n.$$

EOMs

$$\partial_t T^{0\mu} + \partial_i T^{i\mu} = -\Gamma^\nu_{\nu 0} T^{0\mu} - \Gamma^\mu_{0\nu} T^{0\nu} - \Gamma^\nu_{\nu i} T^{i\mu} - \Gamma^\mu_{i\nu} T^{i\nu},$$
$$\partial_t X_i^A - \partial_i Y^A = 0,$$
$$\partial_t Z_i - \partial_i S_n = 0,$$
$$\partial_t U^A = Y^A,$$
$$\partial_t n = S_n.$$

Auxiliary Definitions

$$Y^A = \tilde{P}^{A\lambda} M^{-1}_{\lambda\mu}\left[T^{0\mu} - T_0^{0\mu} - H^{\mu 0 k\beta} P_{B\beta} X_k^B - 2\beta_n u^{(\mu}\Omega^{k0)} Z_k - H^{\mu 0\alpha\beta} C_{k\beta}\Gamma^k_{\alpha\lambda} u^\lambda\right],$$
$$T^{ij} = T_0^{ij} + H^{ij0\beta} P_{A\beta} Y^A + H^{ijk\beta} P_{A\beta} X_k^A + 2\beta_n u^{(i}\Omega^{kj)} Z_k + H^{ij\alpha\beta} C_{k\beta}\Gamma^k_{\alpha\lambda} u^\lambda.$$

Symbolic Expressions for Code

BDNK Tensor

$$T^{\mu\nu} = T_0^{\mu\nu} + H^{\mu\nu 0\beta} P_{A\beta} Y^A + H^{\mu\nu j\beta} P_{A\beta} X_j^A + 2\beta_n u^{(\mu}\Omega^{j\nu)} Z_j + H^{\mu\nu\alpha\beta} C_{j\beta} \Gamma^j_{\alpha\lambda} u^\lambda$$

$$Y^A = \tilde{P}^{A\lambda} M^{-1}_{\lambda\mu} \left[T^{0\mu} - T_0^{0\mu} - H^{\mu 0 k\beta} P_{B\beta} X_k^B - 2\beta_n u^{(\mu}\Omega^{k0)} Z_k - H^{\mu 0\alpha\beta} C_{k\beta} \Gamma^k_{\alpha\lambda} u^\lambda \right], \qquad \sim 10^5$$

$$T^{ij} = T_0^{ij} + H^{ij0\beta} P_{A\beta} Y^A + H^{ijk\beta} P_{A\beta} X_k^A + 2\beta_n u^{(i}\Omega^{kj)} Z_k + H^{ij\alpha\beta} C_{k\beta} \Gamma^k_{\alpha\lambda} u^\lambda. \qquad \sim 10^3$$

- Large number of terms via contractions.
- xAct symbolic simplifications reduces this significantly

Simulating BDNK

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$
$$\nabla_\mu T^{\mu\nu} = 0$$

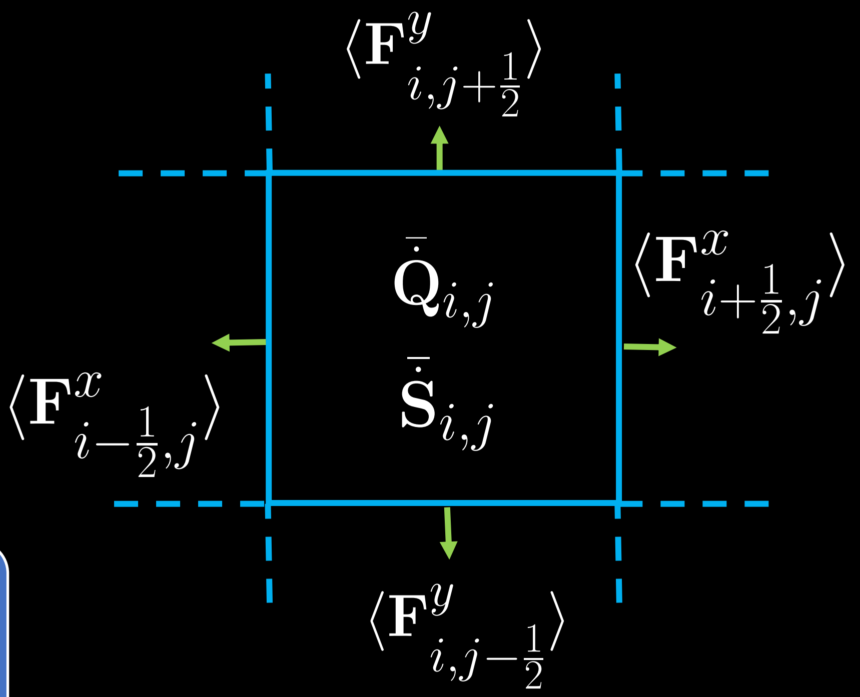
- BDNK EOMs are second order

- Finite Volume Method
  - Equations cast in a First Order Flux Conservative Form.
  - Volume is divided into cells.
  - Discretized variables are cell-averaged and derivatives have to be computed on interfaces.

$$\partial_t \mathbf{Q} + \partial_i \mathbf{F_i} = \mathbf{S}$$

Variable Flux Sources

$$\dot{\bar{\mathbf{Q}}}_{i,j} + \frac{\langle \mathbf{F}^x_{i+\frac{1}{2},j}\rangle - \langle \mathbf{F}^x_{i-\frac{1}{2},j}\rangle}{\Delta x} + ..\text{“}y\,term\text{”}\ldots = \bar{\mathbf{S}}_{i,j}$$



There is a plethora of numerical techniques to compute the derivatives on cell interfaces

Simulating BDNK

$$\partial_t \mathbf{Q} + \partial_i \mathbf{F_i} = \mathbf{S}$$

Variable

Flux

Sources

First Order

New variables lead to new constraint equations

Numerically, they are violated

Need additional auxiliary variables and extra EOMs

$$Y^1 \equiv \partial_t U^1 \qquad X_i^1 \equiv \partial_i U^1$$

$$\partial_t X_i^A - \partial_i Y^A = 0$$

$$\partial_t X_i^1 + \epsilon_{ijk}\partial_j \varphi_k^1 - \partial_i Y^1 = 0$$

$$\partial_t \varphi_i^1 - \kappa \epsilon_{ijk}\partial_j X_k^1 + \partial_i \phi^1 = -\beta \varphi_i^1$$

$$\{T^{0\mu}, X_i^A, \log\epsilon, U^i, n, \varphi_i^A, \phi^A, \Psi_i, \psi\}$$

Hybrid

Derivatives are numerically evaluated

Not exactly flux-conservative

Fewer variables : Computationally cheaper

$$\{T^{0\mu}, \log\epsilon, U^i, n\}$$

Relativistic Hydrodynamics

Euler Equations

In relativity, u^μ
we have 4-velocity
$\mu = \{0, 1, 2, 3\}$
$\{t, x, y, z\}$
Normalization $(c = 1)$
$u^\mu u_\mu = -1$

Conservation of energy-momentum

$$\dot{\vec{u}} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla P \qquad \nabla_\mu T^{\mu\nu} = 0$$

Conservation of mass becomes conservation of 4-current

$$\dot{\rho} + \vec{u} \cdot \nabla\rho = 0 \qquad \nabla_\mu J^\mu = 0$$

Ideal Fluid Energy-Momentum Tensor

$$T_0^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu}$$

4-Current

$$J^\mu = \rho u^\mu$$

Rest Frame

$$u^\mu = (1, 0, 0, 0)$$

$$J^\mu = (\rho, 0, 0, 0)$$

$$\begin{pmatrix} e & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

$$T^{\mu\nu} \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}$$

Metric Tensor

$$g^{\mu\nu}$$

Einstein's equations : Energy-Momentum $\leftrightarrow$ Spacetime Geometry

Shock smoothing