



Robust 3+1D simulations of BDNK causal relativistic viscous hydrodynamics

Toward dissipative relativistic astrophysics

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VandyGRAF (Vanderbilt initiative for gravity, waves, and fluids)

arXiv:26??.xxxx Ongoing work with (Hirvonen H., Disconzi M., Bemfica F.S., Paquet JF, Noronha J.)

arXiv:2510.1660 (with Clarisse, N., Pinho, E. O., Patel, T., Bemfica, F. S., Hippert, M., & Noronha, J.)



U.S. DEPARTMENT
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Relativistic Hydrodynamics

Effective theory for long-wavelength fluid dynamics
Fluid variables: Velocity, Density and Pressure

Ideal (Inviscid)
Fluids
Non-dissipative

Non-Relativistic

$$\dot{\vec{u}} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla P$$

Relativistic

$$\nabla_{\mu} T^{\mu\nu} = 0$$

4 DOFS

$$T_0^{\mu\nu} = (\epsilon + P) u^{\mu} u^{\nu} + P g^{\mu\nu}$$

Energy-Momentum Tensor

Foundation of GRHD/GRMHD simulations in relativistic astrophysics

Ideal Relativistic Hydrodynamics + Einstein equations + Maxwell equations

Viscous
Fluids
Dissipative

Navier-Stokes

$$\dot{\vec{u}} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \vec{u}$$

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$

IDEAL

Viscous Terms

Relativistic Viscous Formulations

Recent interest in astrophysics

Most et al. '21

Chabanov & Rezzolla '23 BNS mergers

Espino et al. '23

Shibata et al. '25 Accretion

Heavy-ion collisions / Quark Gluon Plasma

*Established application of relativistic
viscous hydrodynamics*



$$\dot{\vec{u}} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \vec{u}$$

How do we formulate relativistic viscosity?

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$

IDEAL

Viscous Terms

Relativistic Generalization of Navier-Stokes

- Gradient Expansion: Eckart (1940) and Landau-Lifshitz (1950s).
- Acausal and Unstable.

Causality + Stability + Hyperbolicity
Essential for relativity and numerical simulations

**First-Order theories are always
acausal and unstable?**

BDNK: Causal First-order Viscous Hydrodynamics

Causal and Stable First-Order Gradient Expansion Theory

IDEAL

$$T_0^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$

BDNK (**B**emfica, **D**isconzi, **N**oronha, **K**ovtun)

Bemfica, Disconzi and Noronha, 2018
Kovtun, 2019

Most general viscous energy-momentum tensor with first order gradient terms.

$$T_{\mu\nu} = (\epsilon + \mathcal{A})u_\mu u_\nu + (P + \Pi)\Delta_{\mu\nu} - 2\eta\sigma_{\mu\nu} + u_\mu \mathcal{Q}_\nu + u_\nu \mathcal{Q}_\mu$$

Example: Gradient correction to the energy density

$$\mathcal{A} = \tau_\epsilon [u^\mu \nabla_\mu \epsilon + (\epsilon + P) \nabla_\mu u^\mu]$$

Explicit constraints on transport coefficients $\longleftrightarrow$ CAUSAL | STABLE | HYPERBOLIC

Causality & Stability conditions: Can be checked when initializing

Can we turn it into a useful general numerical tool?

Simulating BDNK

2+1D Flat Spacetimes

Pandya et al. '21, '22

2+1D QGP: Heavy Ion Collisions

Bantilan et al. 2022, Bea et al. 2023,
Bhambure et al. 2025, Fantini et al. 2025

1+1D Neutron Stars

Keeble et al. 2025,
Shumet et al. 2025.

Our Work : General 3+1 Dimensional Relativistic Simulations

Numerical Implementation :

- Arbitrary Spacetimes (Thousands of terms)
- General Equation of State (Analytical and Tabulated)
- C++ Portable code  **kokkos** (Scalable on CPUs and GPU clusters)

Testing is crucial!

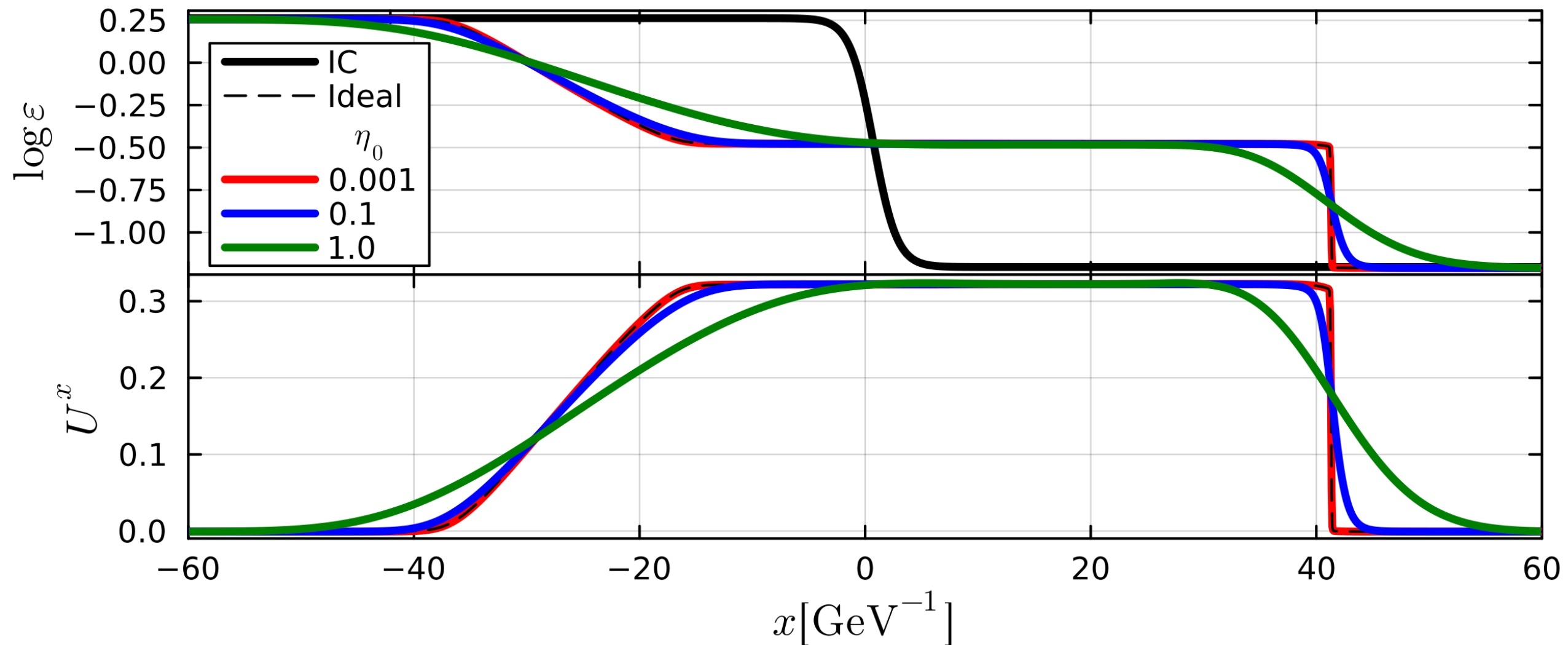
Shock tests

Convergence tests

Low viscosity ↔ Ideal Hydro

Validation I: Shock Tube tests

- Validated against code of Clarisse et al (2025)
 - Agreement with Ideal at low viscosity



Validation II: Gubser Flow

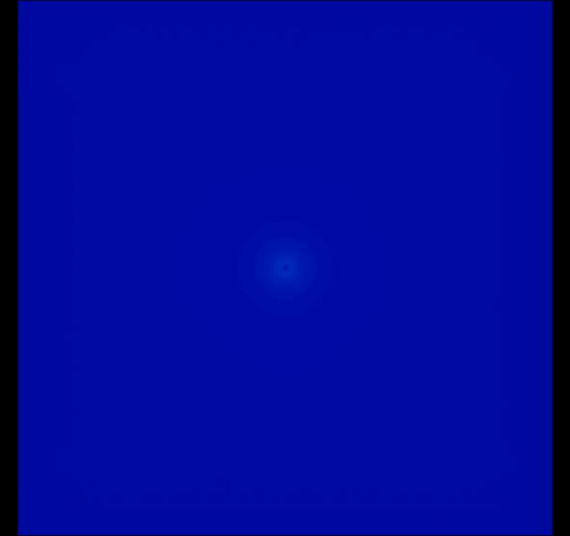
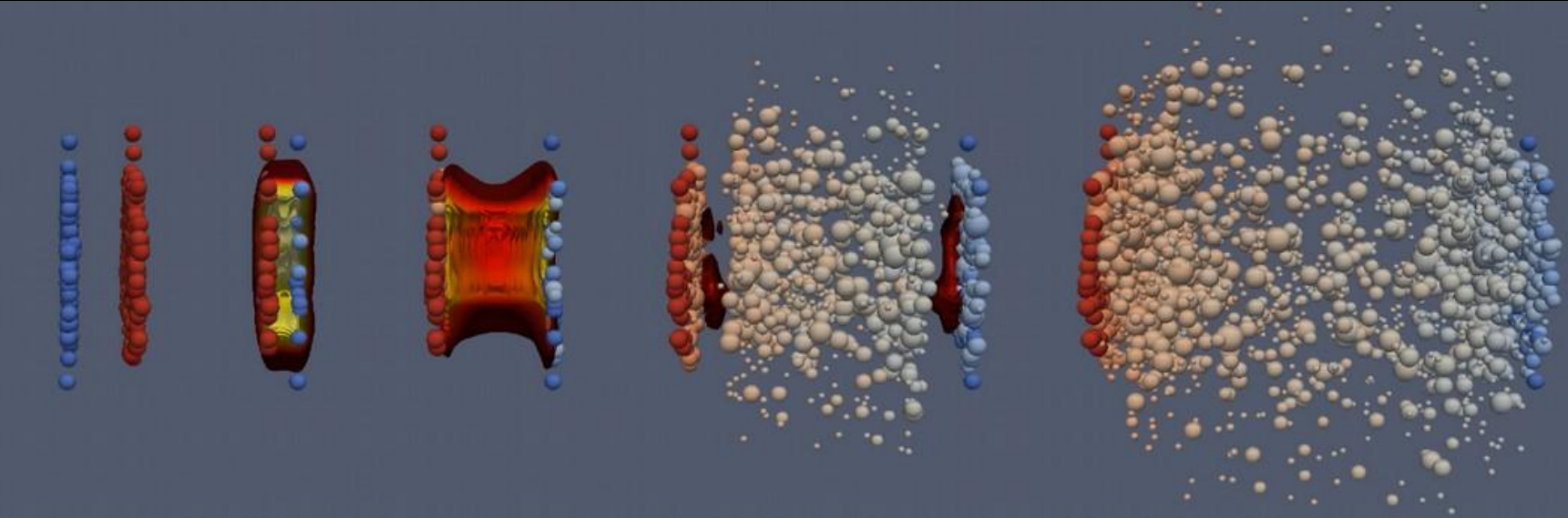


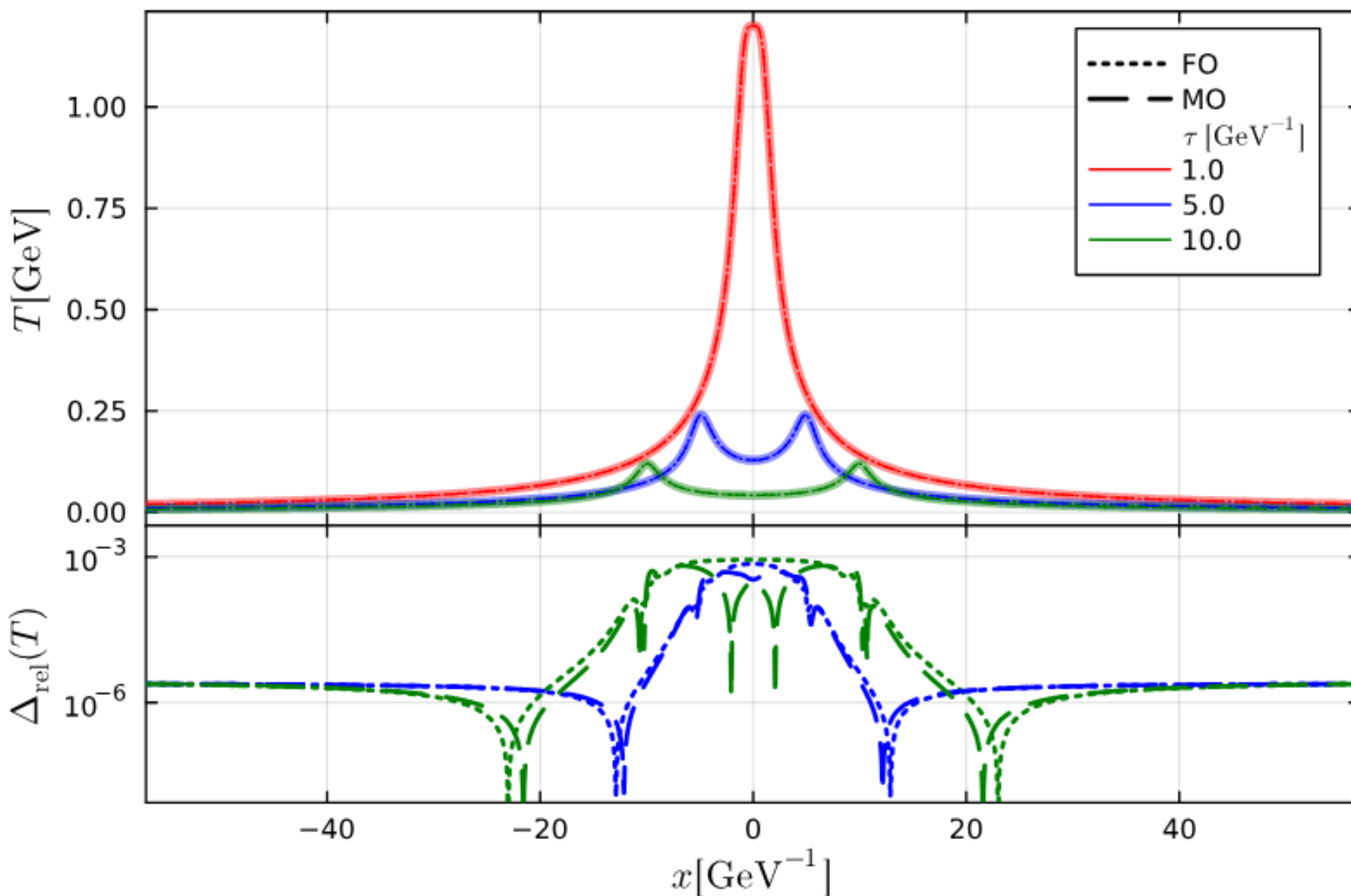
Illustration of a Heavy Ion Collision

[Ref: MADAI collaboration, Hannah Petersen and Jonah Bernhard]

Gubser Flow

- Cylindrically symmetric flow with semi-analytical solution
 - Toy model for QGP expansion
 - Semi-analytical solution : Bemfica et al. 2018

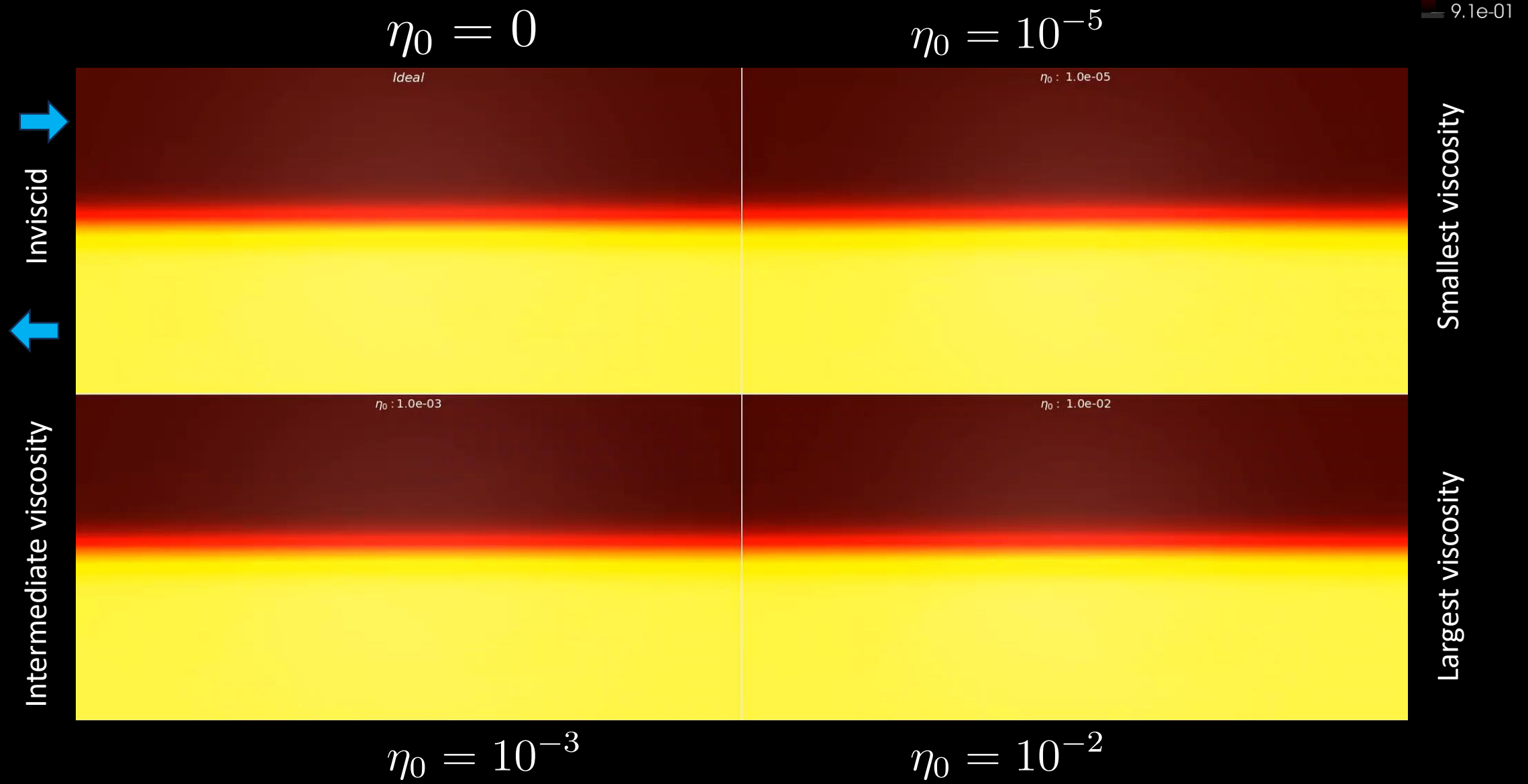
Validation II: Gubser



- Conformal fluid in 2+1D
- Milne metric: Hyperbolic coordinates used in heavy ion collisions
- Initialized with semi-analytical solution.
- Agreement with semi-analytical solution.
- Two formalisms are consistent.

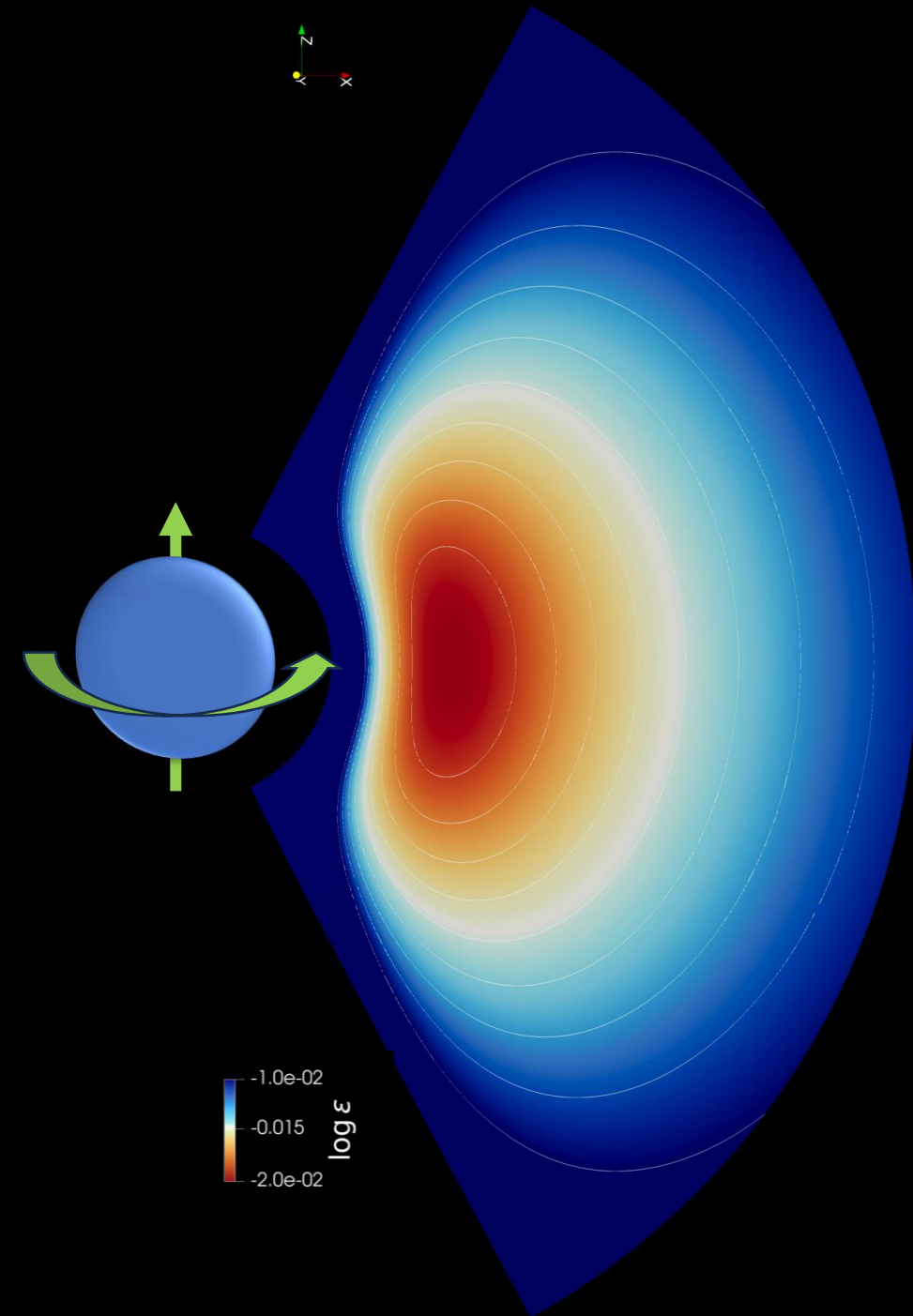
Validation III (Ideal Recovery): Kelvin Helmholtz

Low viscosity BDNK matches Ideal Hydro solution

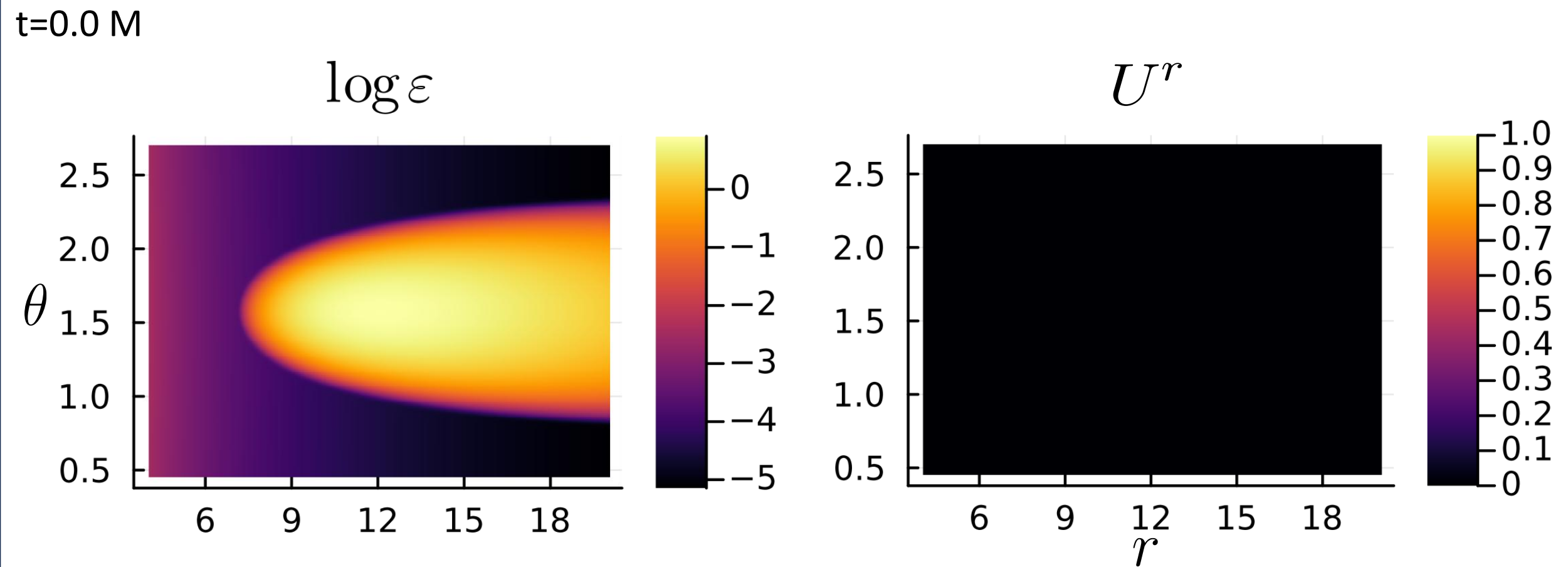


¹⁰Validation IV : Curved Spacetime Kerr Torus

- Fishborne-Moncrief stationary disk:
 - Known Ideal Hydro stationary solution for purely azimuthal flow in Kerr. $u_\mu = u_t (1, 0, 0, -l)$
- Used in testing ideal relativistic hydrodynamics codes
- Our BDNK implementation
 - Near-Extremal $a/M=0.99$
 - Challenges: Boundary effects, sharp discontinuities, atmosphere
 - Critically, the ideal solution is not a stationary solution in BDNK!



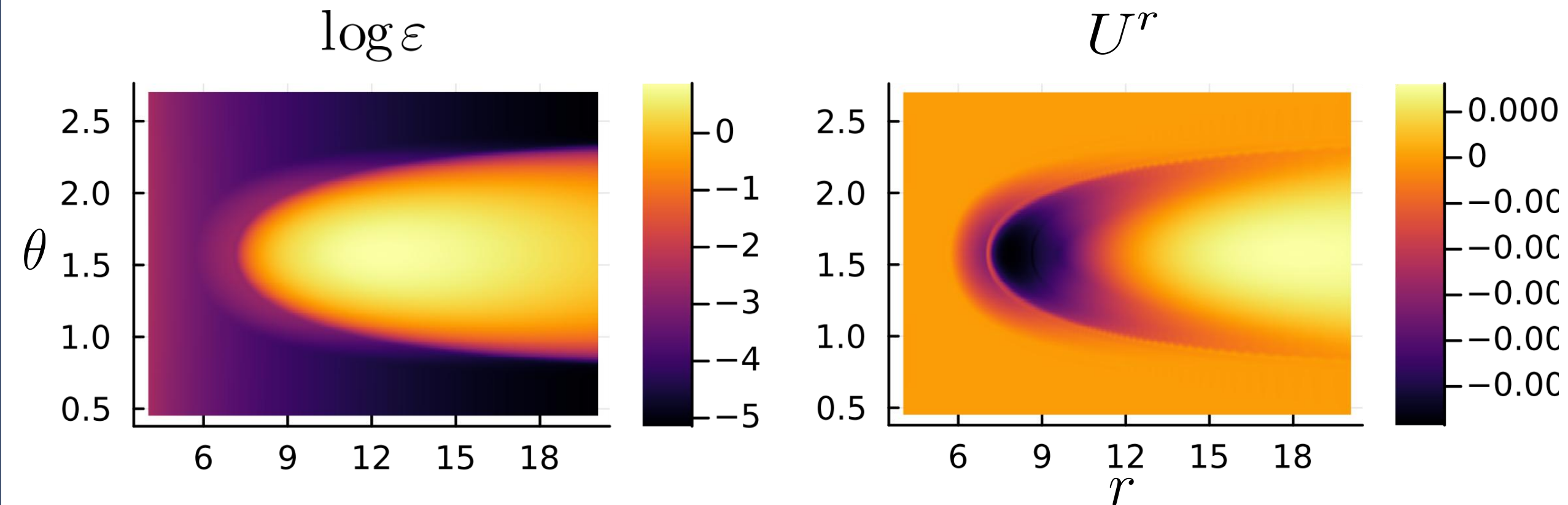
Validation IV : Curved Spacetime – Kerr Torus



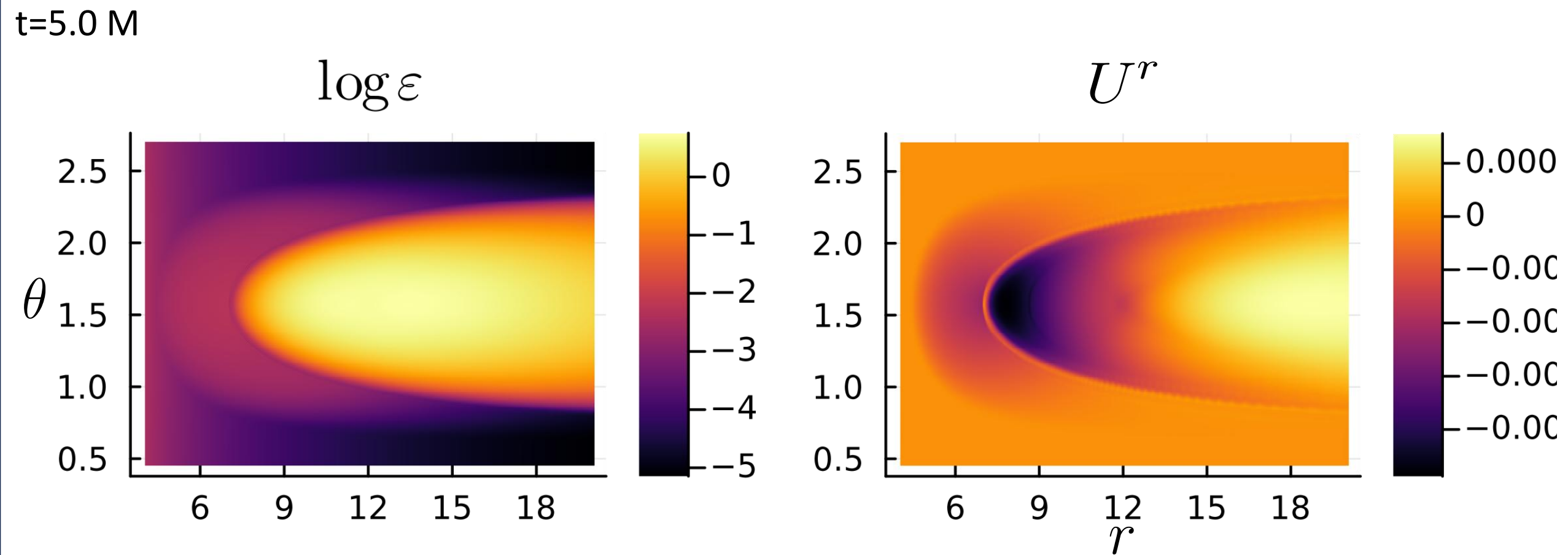
- Initial conditions: Ideal stationary Fishborne-Moncrief disk
- In BDNK, small but finite viscosity leads to infall

Validation IV : Curved Spacetime – Kerr Torus

t=2.5 M



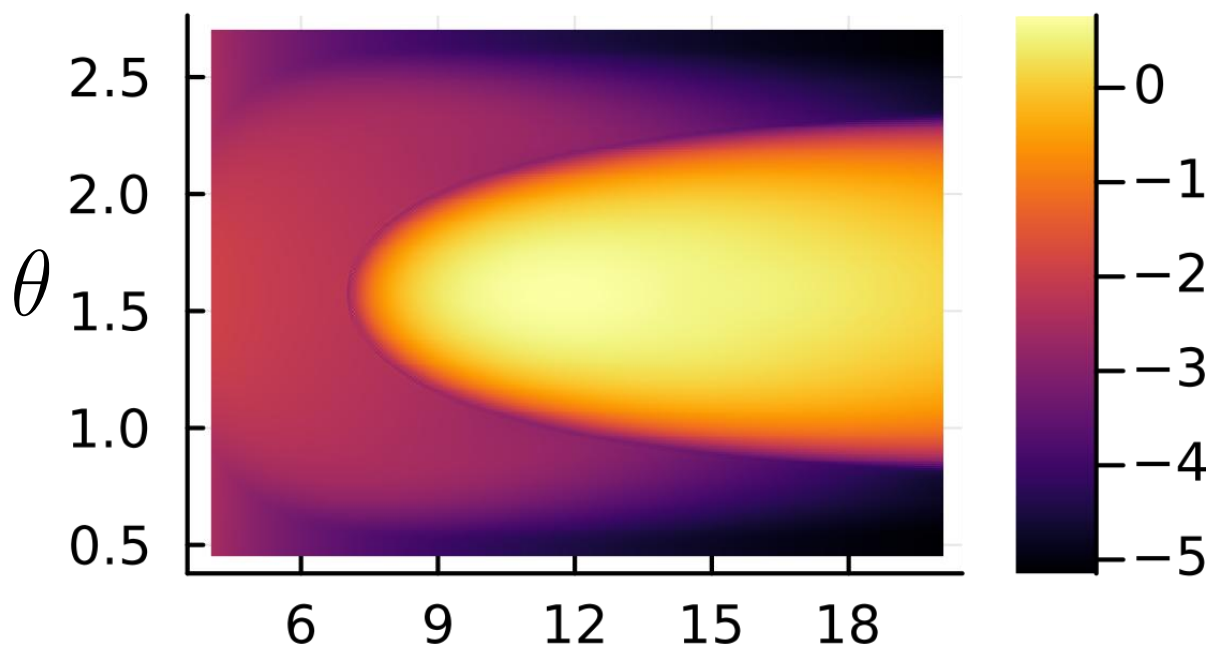
Validation IV : Curved Spacetime – Kerr Torus



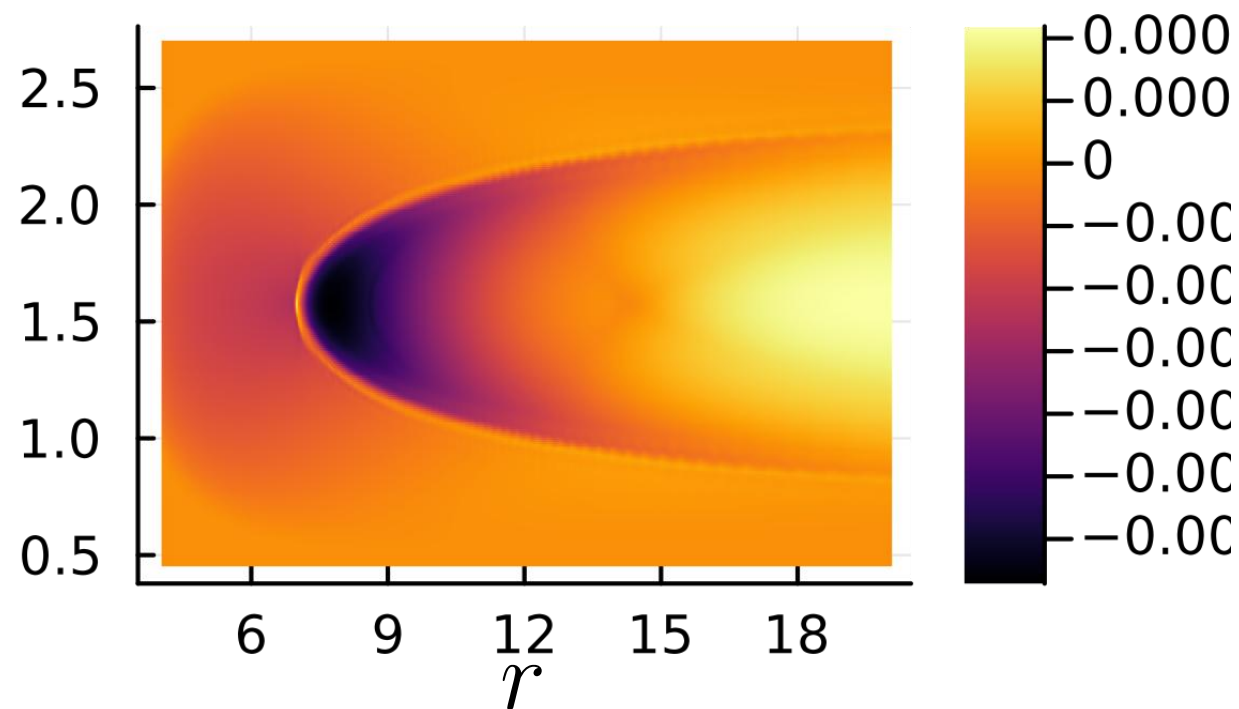
Validation IV : Curved Spacetime – Kerr Torus

$t=7.5 M$

$\log \varepsilon$

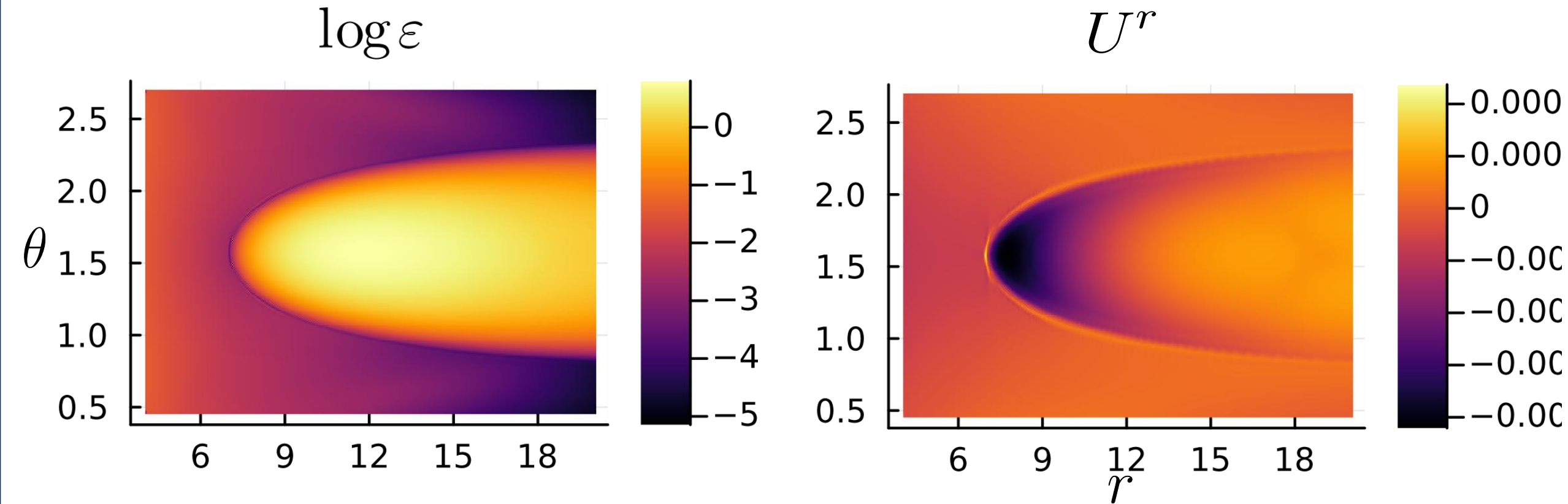


U^r



Validation IV : Curved Spacetime – Kerr Torus

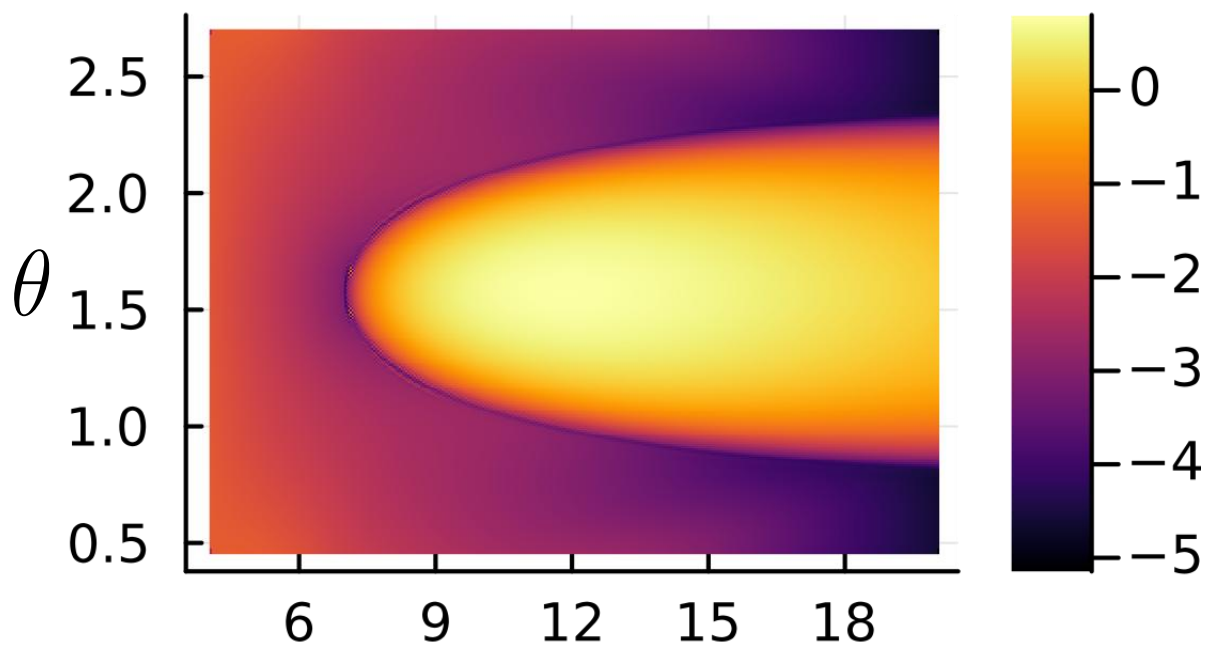
t=10.0 M



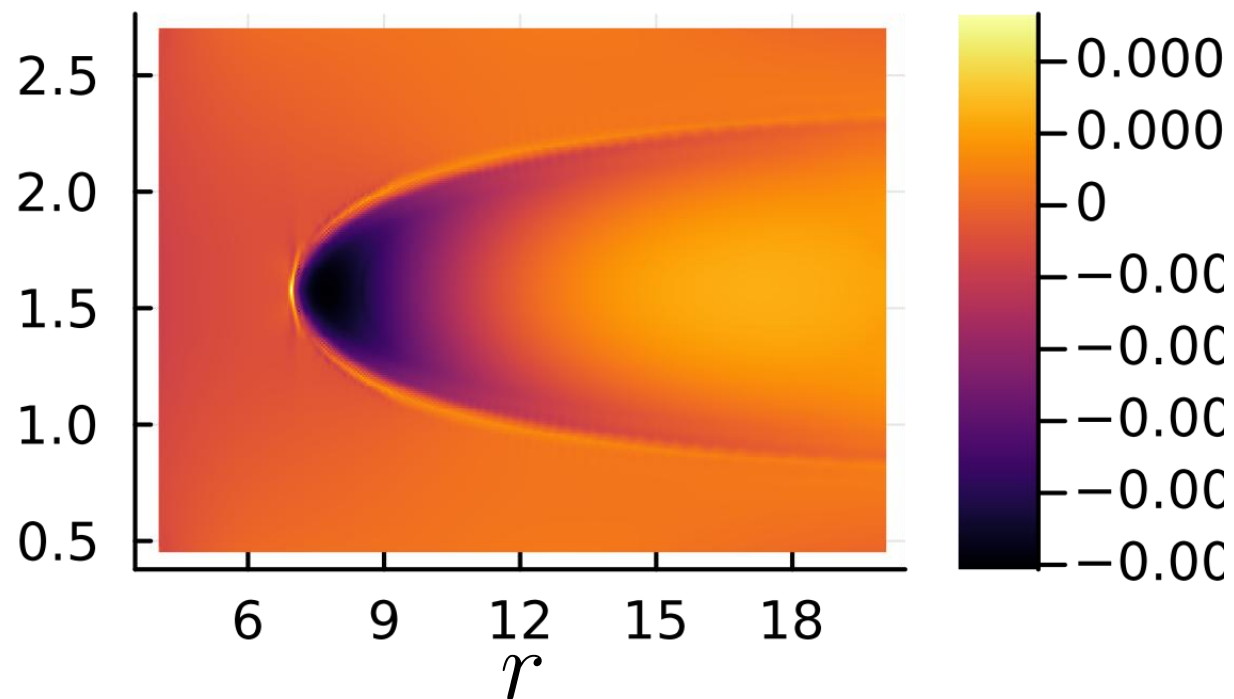
Validation IV : Curved Spacetime – Kerr Torus

t=12.5 M

$\log \varepsilon$

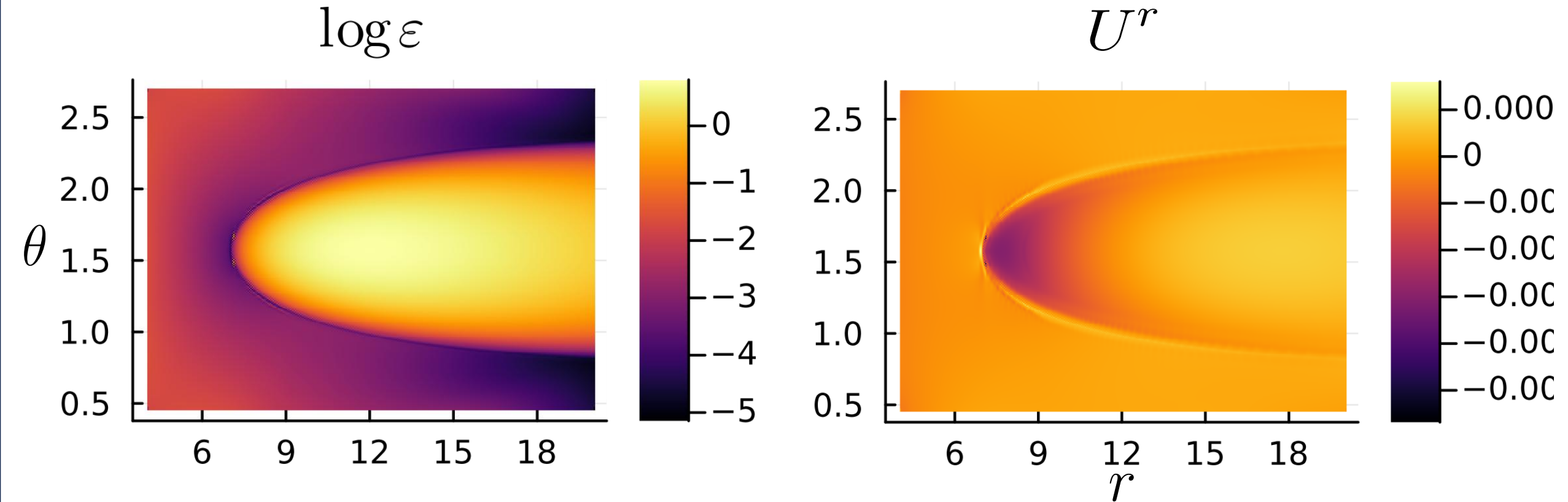


U^r



Validation IV : Curved Spacetime – Kerr Torus

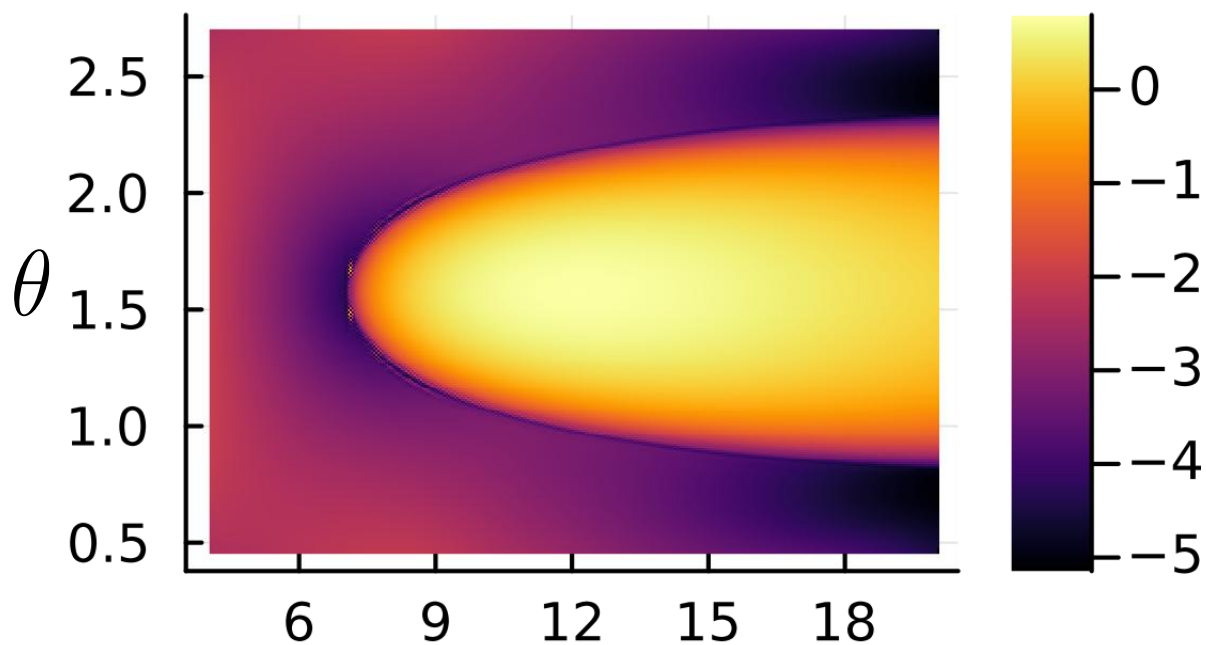
t=15.0 M



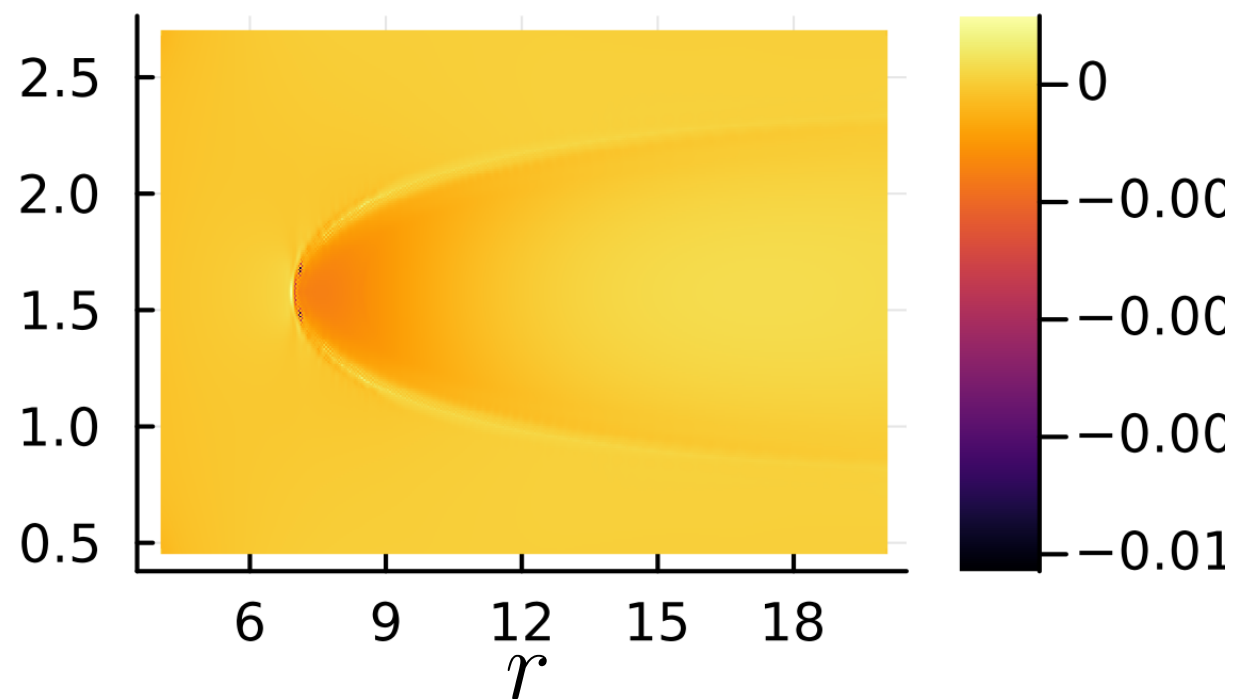
Validation IV : Curved Spacetime – Kerr Torus

t=20.0 M

$\log \varepsilon$



U^r



Summary

- BDNK : Causal and Stable Relativistic Viscous Hydrodynamics
- 3+1D, general spacetimes, arbitrary EOS
- Validation
 - Reproduced known semi-analytical solutions
 - Low-Viscosity agreement with Ideal-Hydro solutions
 - Comparisons with Israel-Stewart theory for QCD EOS
 - Shock Tube tests

arXiv:26??.xxxx

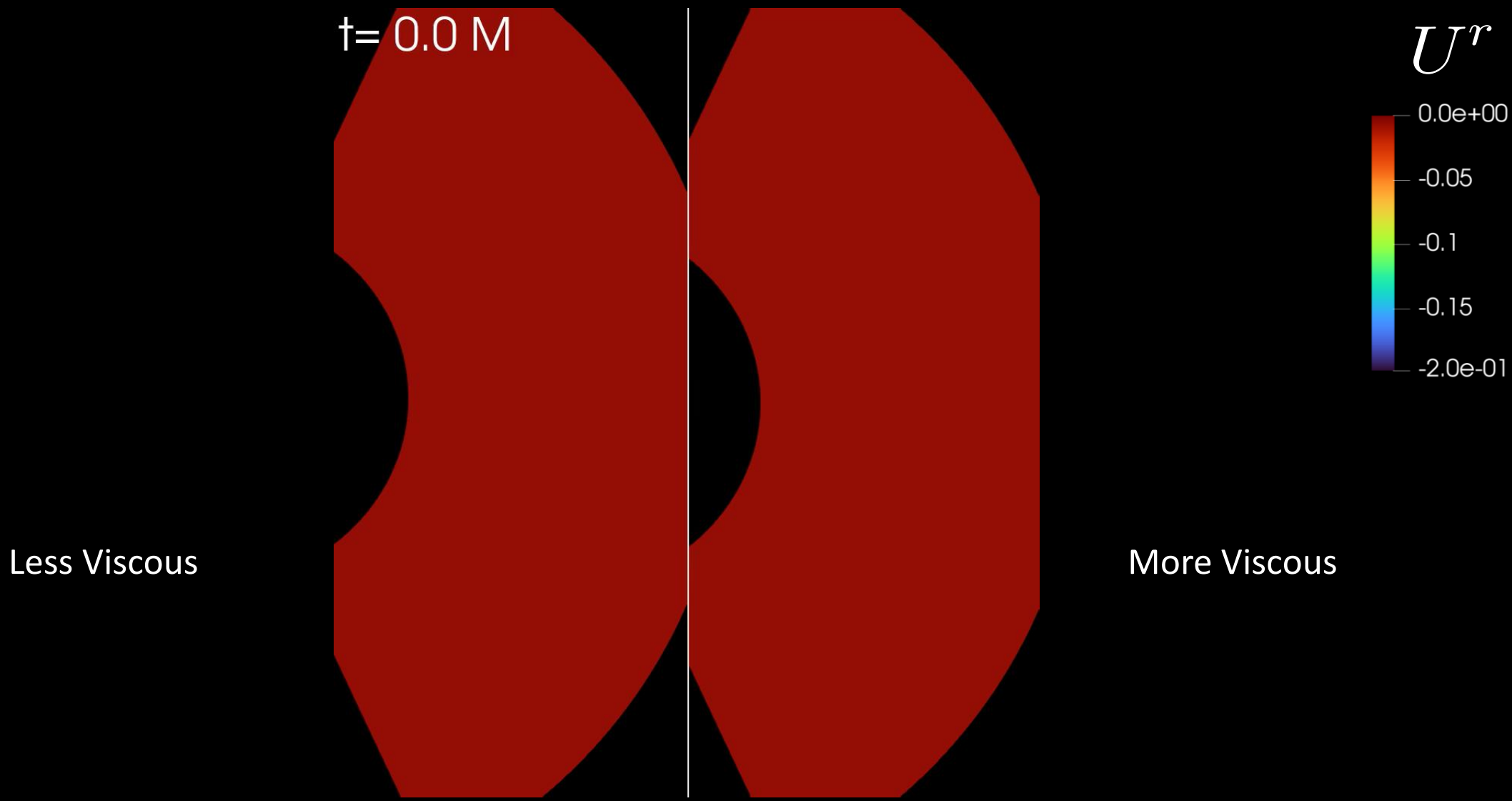
Outlook

- Open Source Code (HIC and Astrophysics community)
- Applications to Heavy Ion Collision data
- AMR for Large Scale GR-Hydro Simulations
- Visco-resistive MHD formalism
- 2+1D, 3+1D Shock Analysis

Lots of 'exciting' technical challenges remain on the mathematical formulation as well practical implementation and applications!

Validation V : Curved Spacetime – Kerr Disks

Comparing viscous effects



Summarizing the viscous landscape

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$

IDEAL

$$T_0^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$

Relativistic Navier Stokes

$$\dot{\vec{u}} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \vec{u}$$

$$T_{\mu\nu} = (\epsilon + \mathcal{A}) u_\mu u_\nu + (P + \Pi) \Delta_{\mu\nu} - 2\eta \sigma_{\mu\nu} + u_\mu Q_\nu + u_\nu Q_\mu$$

	CAUSAL	STABLE	HYPERBOLIC	CODES
Landau/Eckart: Bad Frame Choice $\mathcal{A} = 0$ OR $Q^\mu = 0$				
BDNK: General choice $\mathcal{A} = \tau_\epsilon [u^\mu \nabla_\mu \epsilon + (\epsilon + P) \nabla_\mu u^\mu]$ Explicit algebraic constraints on transport coefficients				THIS WORK
Israel-Stewart theories • 10 New degrees of freedom • Causality conditions only partially known • Exceedingly complicated GR implementation	 			

Backups

Simulating BDNK

2+1D Flat Spacetimes

Pandya et al. '21, '22

2+1D QGP: Heavy Ion Collisions

Bantilan et al. 2022, Bea et al. 2023,
Bhambure et al. 2025, Fantini et al. 2025

1+1D Neutron Stars

Keeble et al. 2025,
Shumet et al. 2025.

Our Work : General 3+1 Dimensional Relativistic Simulations

Mathematical Formulation (2 Mathematically Equivalent Formulisms):

- Full First order formulism (Strictly Flux Conservative with Curl-Cleaning)
- Hybrid formulism (Computationally cheaper – Similar to HIC sims)

Numerical Implementation :

- Arbitrary Spacetimes (Thousands of terms)
- General Equation of State (Analytical and Tabulated)
- C++ Portable code  **kokkos** (Scalable on CPUs and GPU clusters)

Testing is crucial!

Convergence tests

Shock tests

Low
viscosity

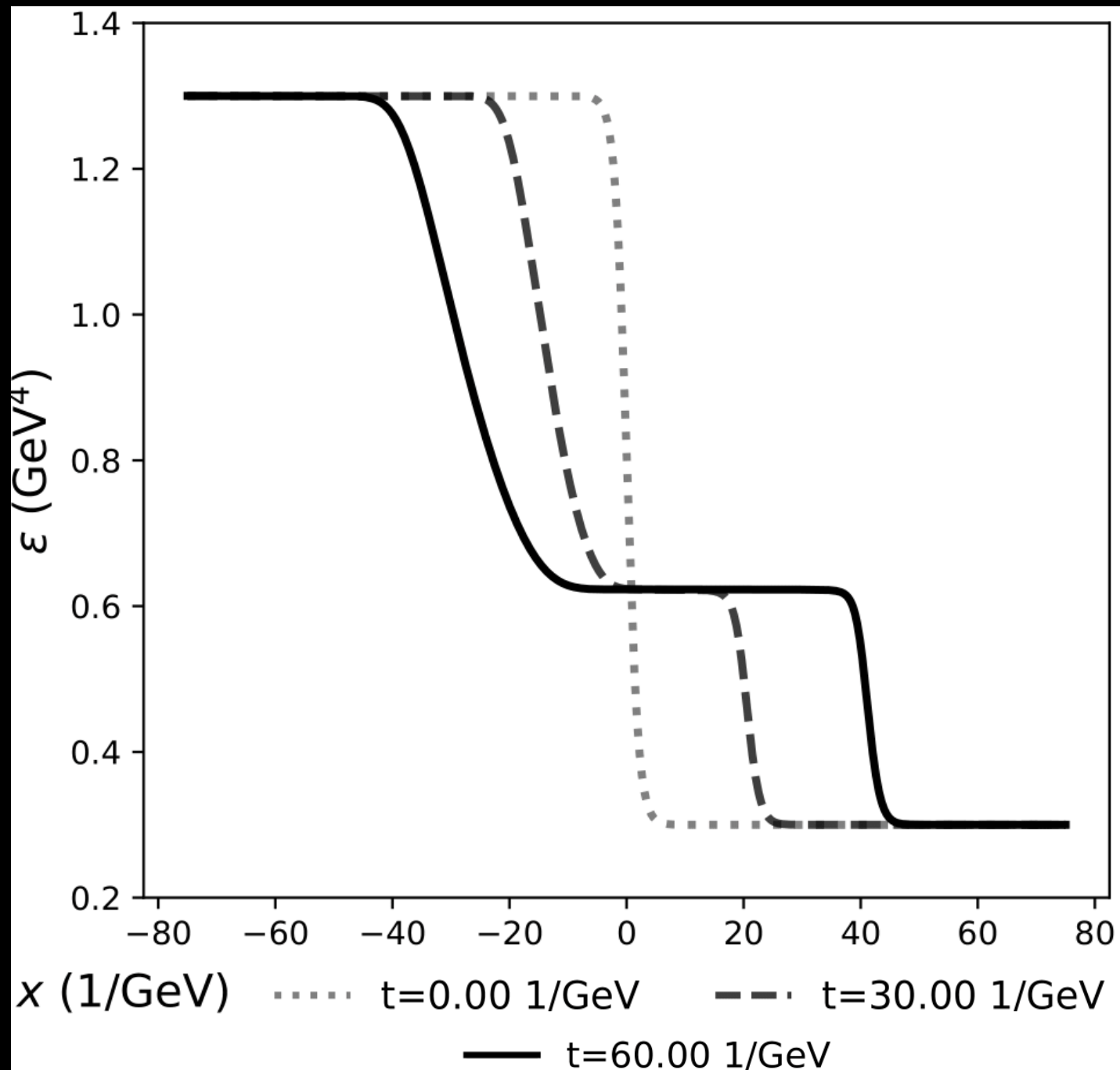


Ideal
Hydro

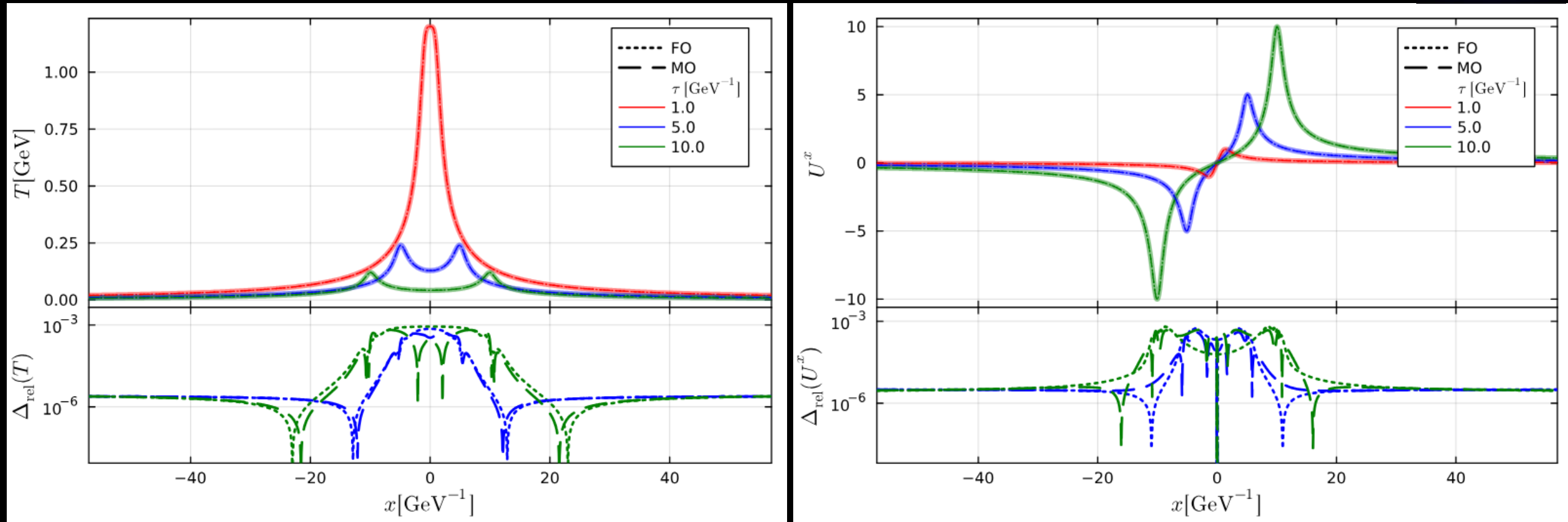
Validation I: Shock Tube tests

Shock tests conducted to check numerical stability : With an independent code

Clarisse, N., Pinho, E. O., Patel, T., Bemfica, F. S., Hippert, M., & Noronha, J. (2025).
arXiv:2510.16603.



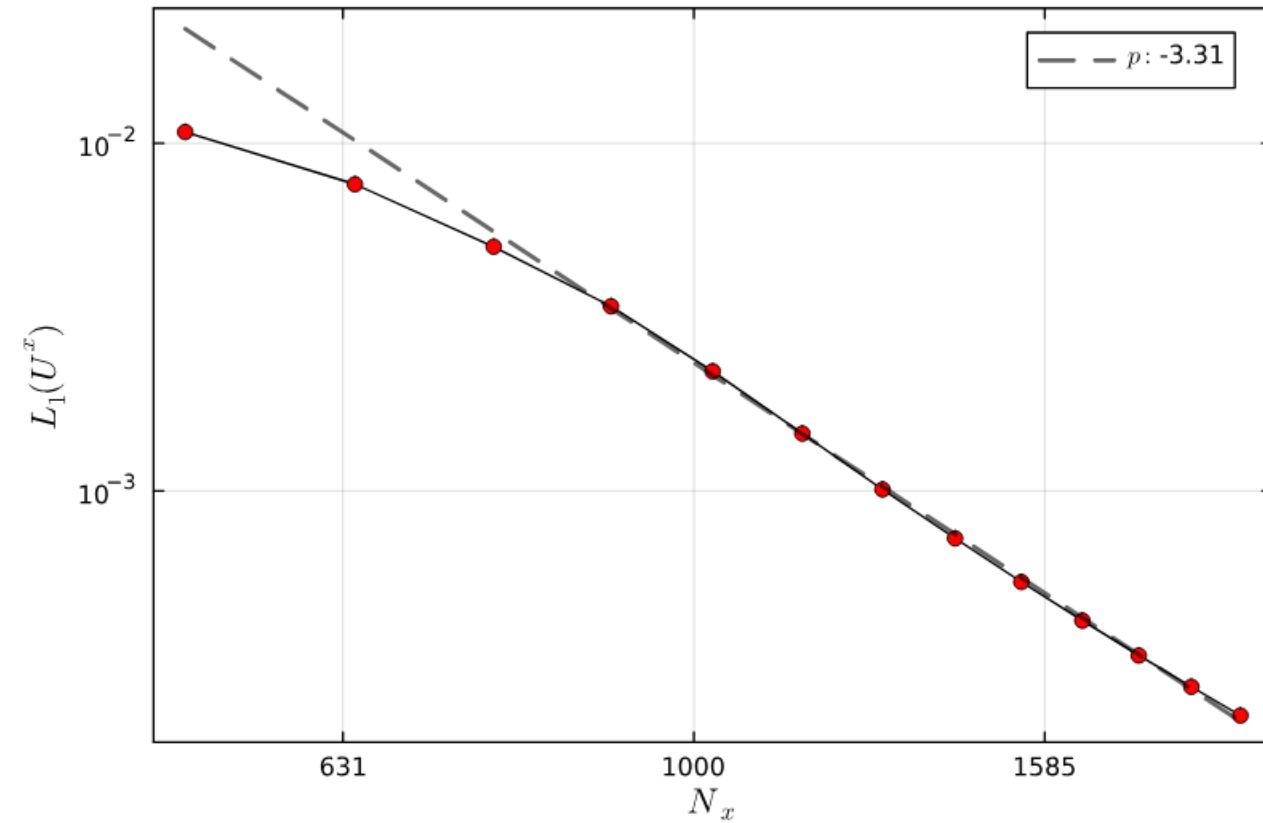
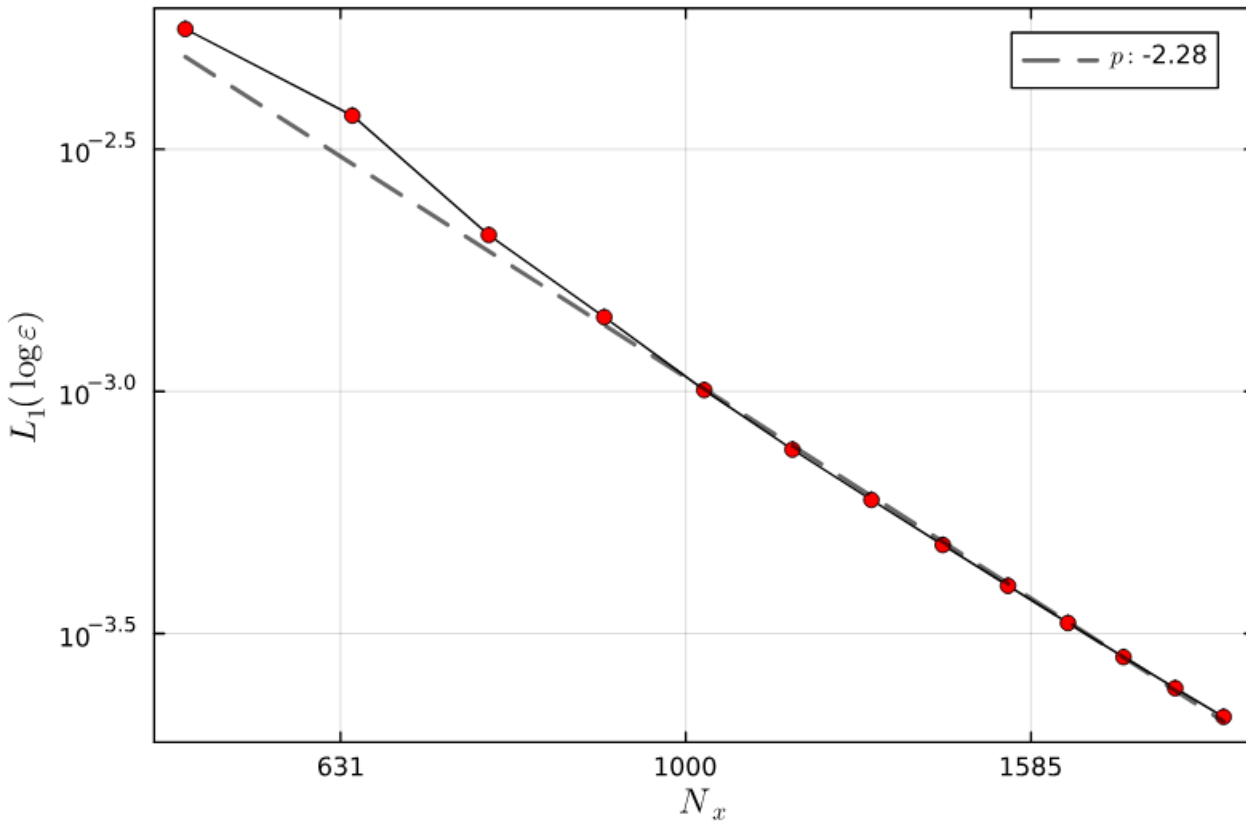
Validation III: Gubser



- Conformal fluid set up in 2+1D (xy-transverse plane).
- Initialized with semi-analytical solution for temperature and velocity.
- Agreement with semi-analytical solution.
- Two formulisms are consistent.

Validation III: Gubser

Convergence Test : Simulation should converge to the continuum solution with resolution
at a certain rate

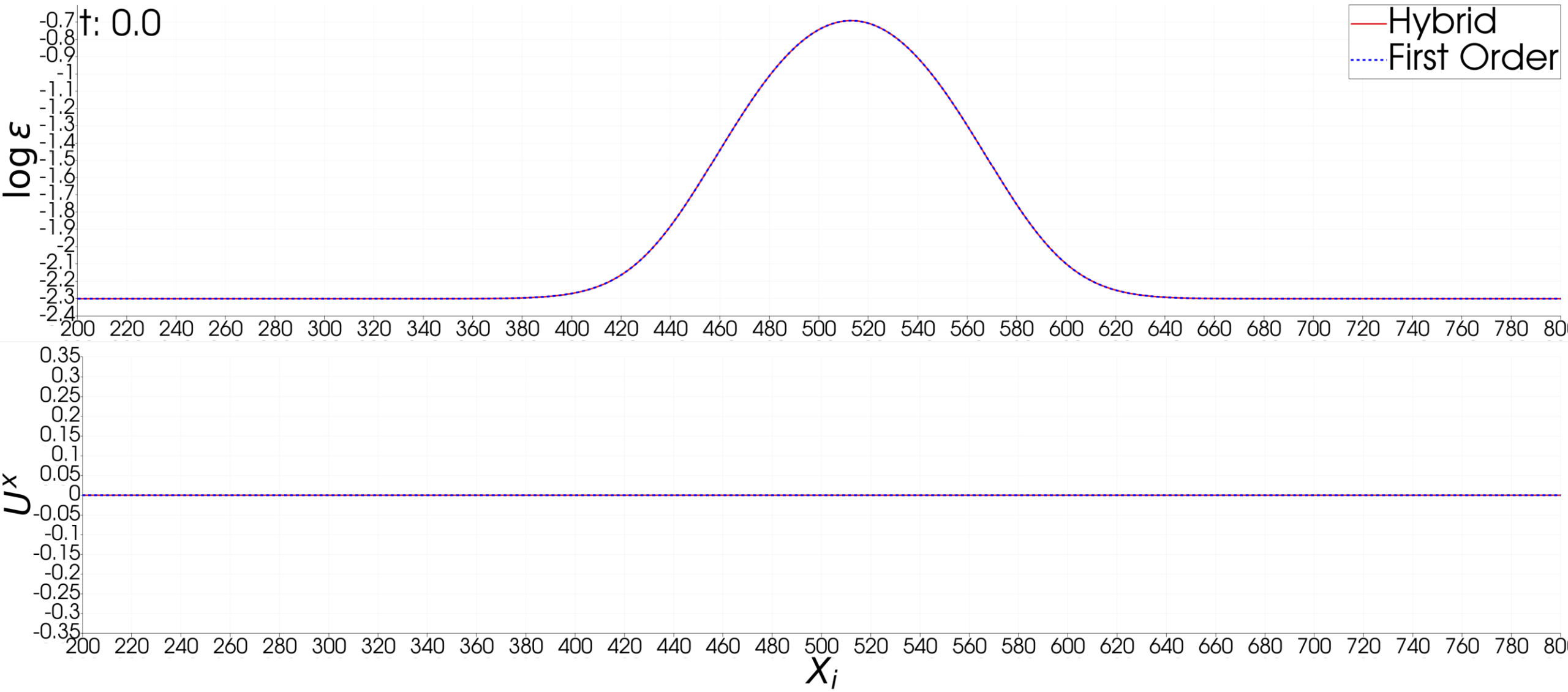


- Error scaling for 2nd Order Finite Volume methods
- Error computed relative to the true semi-analytical solutions

$$\text{Error} \propto \Delta x^{-2}$$

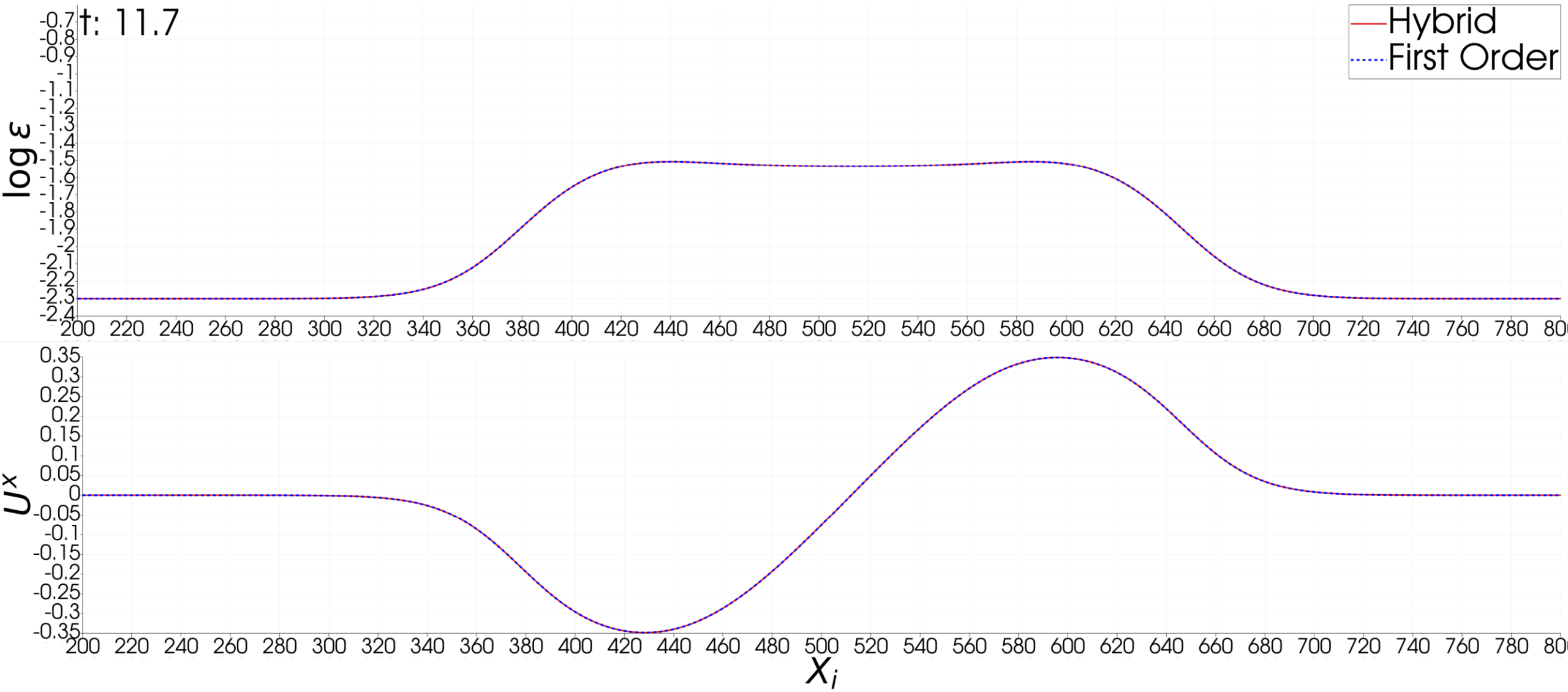
Validation I (2 Formalisms): 1+1 Gaussian

Comparing the results from the two independent formalisms



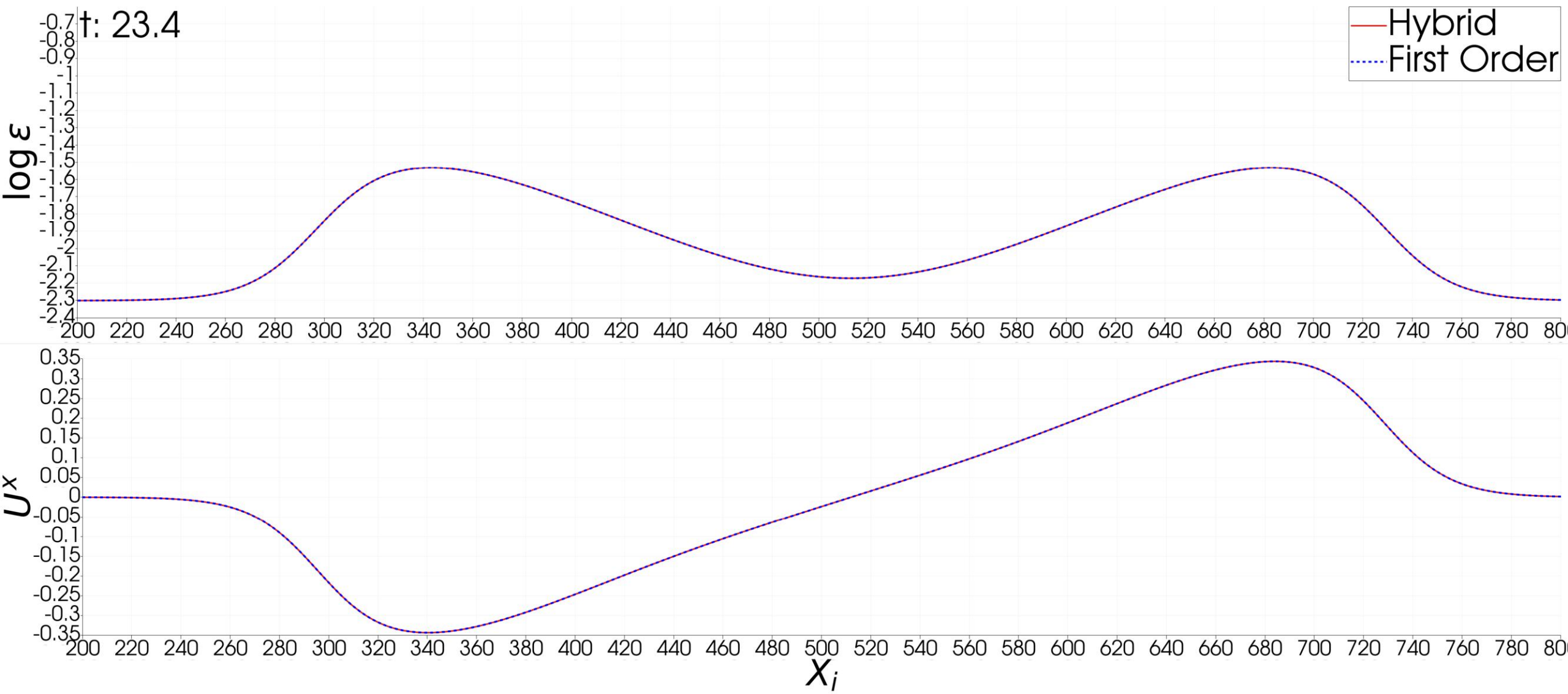
Validation I (2 Formalisms): 1+1 Gaussian

Comparing the results from the two independent formalisms



Validation I (2 Formalisms): 1+1 Gaussian

Comparing the results from the two independent formalisms



Simulating BDNK

Previous Works

- Pandya et al. '21, '22'
- Bea et al. 2023
- Keeble et al. 2025, Shumet et al. 2025.

2+1D : Flat Spacetime
1+1D, 2+1D : QGP
1+1D : Neutron star

Our Work : General 3+1 Dimensional Relativistic Simulations

Finite Volume Method (FVM)

- Extensively used in relativistic and non-relativistic simulations.
- Equations cast in a first-order flux-conservative form.
- Complicated in BDNK : Second order PDE system.
- Our FVM: KT scheme used in MUSIC

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu} \quad 8 \text{ DOFS}$$

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\partial_t \mathbf{Q} + \partial_i \mathbf{F}_i = \mathbf{S}$$

Variable

Flux

Sources

$$Y^1 \equiv \partial_t U^1$$

FIRST ORDER

$$X_i^1 \equiv \partial_i U^1$$

Spatial gradients are treated as independent variables

New variables lead to new constraint equations

Need additional auxiliary variables and extra EOMs

44 Variables $\{T^{0\mu}, X_i^A, \log \epsilon, U^i, n, \varphi_i^A, \phi^A, \Psi_i, \psi\}$

$$Y^1 \equiv \partial_t U^1$$

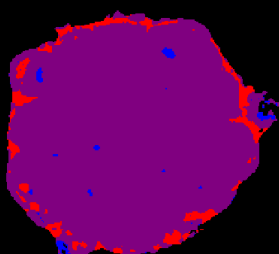
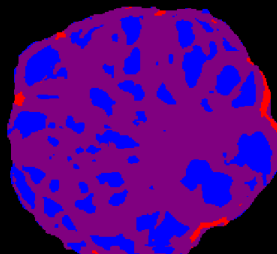
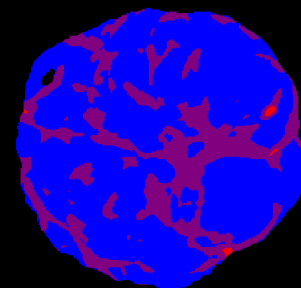
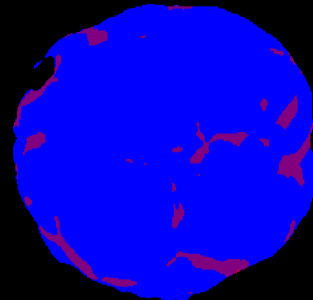
HYBRID / MIXED

Spatial derivatives are numerically evaluated

Not exactly flux-conservative : Sources have derivatives

Fewer variables : Computationally cheaper

9 Variables $\{T^{0\mu}, \log \epsilon, U^i, n\}$

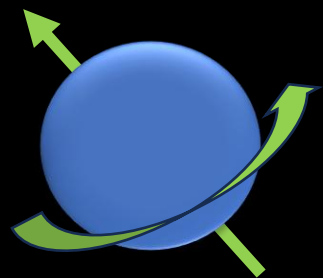
$\tau = 0.40 \text{ fm}/c$  $\tau = 1.40 \text{ fm}/c$  $\tau = 2.40 \text{ fm}/c$  $\tau = 3.40 \text{ fm}/c$  $\tau = 4.40 \text{ fm}/c$ 

IP-Glasma + MUSIC

Plumber et al, 2021

Causality violations in realistic simulations of heavy-ion collisions

Validation V : Curved Spacetime – Kerr Disks

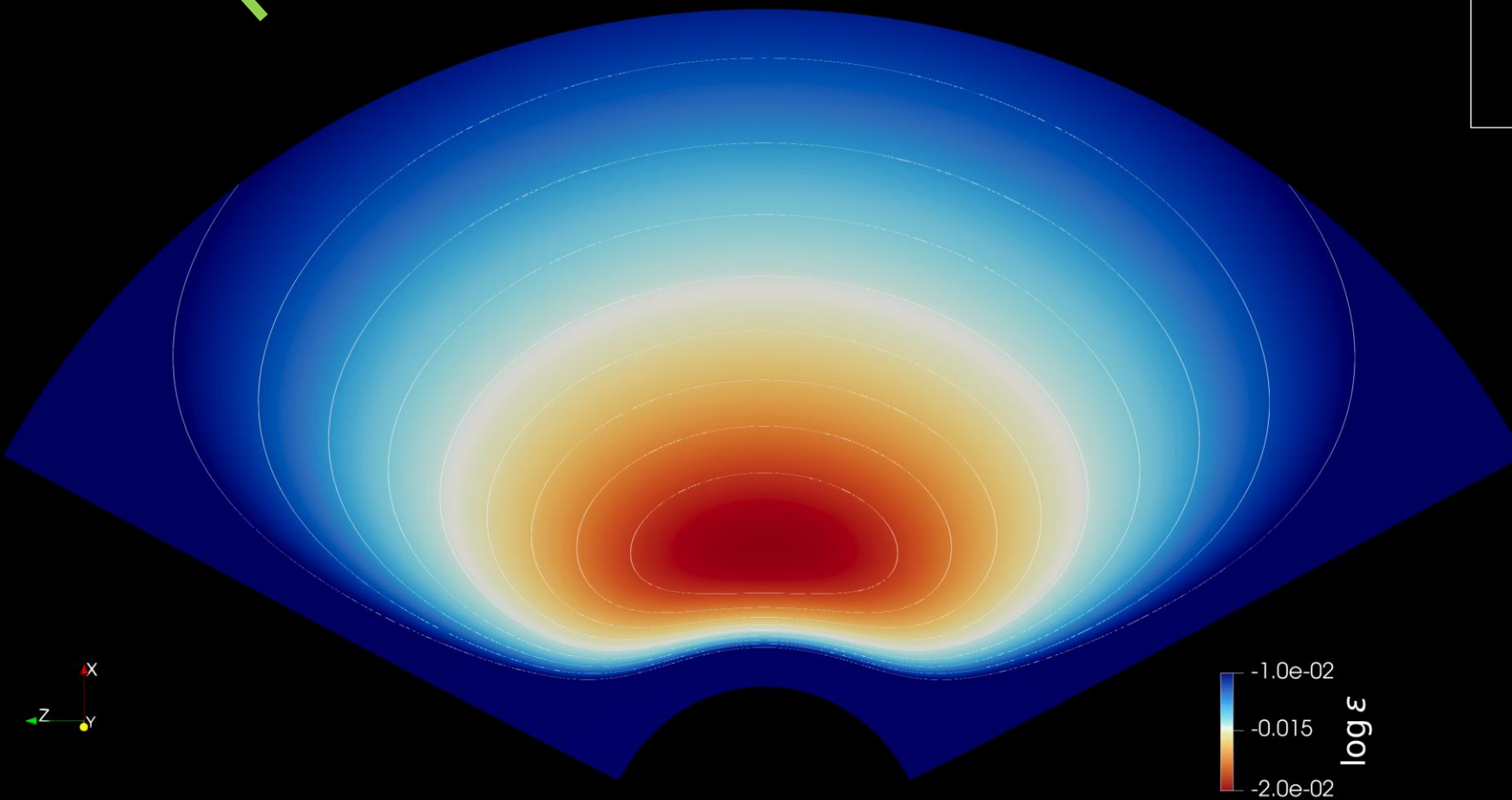


In Boyer–Lindquist coordinates (t, r, θ, ϕ) , the Kerr line element is

$$ds^2 = - \left(1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4Mar \sin^2 \theta}{\Sigma} dt d\phi \\ + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\ + \left(r^2 + a^2 + \frac{2Ma^2r \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2,$$

$$r_{\pm} = M \pm \sqrt{M^2 - a^2} \quad \frac{r_+}{M} \approx 1.141$$

- Ideal Fishborne-Moncrief stationary disk
- Near-Extremal $a/M=0.99$
- Used in testing ideal relativistic hydrodynamics codes
- BDNK implementation challenges: Boundary effects, sharp discontinuities, atmosphere



Definition of out-of-equilibrium variables: Hydrodynamic Frame

$$T_0^{\mu\nu} = (\varepsilon_{eq} + P_{eq}) u_{eq}^\mu u_{eq}^\nu + P_{eq} g^{\mu\nu}$$

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$

- Fictitious T, u^μ
- Ambiguity in parameterizing fluid stress-energy tensor, provided one recovers the unambiguous equilibrium ideal fluid tensor in equilibrium.
- Different choices correspond to different *effective* descriptions.

Minimal IS theory : MUSIC

$$T_{IS}^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$\tau_\Pi u^\mu \partial_\mu \Pi + \Pi = -\zeta \partial_\mu u^\mu$$

$$\tau_\pi u^\sigma \partial_\sigma (\Delta_{\alpha\beta}^{\mu\nu} \pi^{\alpha\beta}) + \pi^{\mu\nu} = -2\eta \sigma^{\mu\nu}$$

BDNK

$$T_{\mu\nu} = (\varepsilon + \mathcal{A}) u_\mu u_\nu + (P + \Pi) \Delta_{\mu\nu} - 2\eta \sigma_{\mu\nu} +$$

$$\mathcal{A} = \tau_\varepsilon [u^\mu \nabla_\mu \varepsilon + (\varepsilon + P) \nabla_\mu u^\mu] = -\tau_\varepsilon \left(u_\alpha \nabla_\beta T_0^{\alpha\beta} \right)$$

Comparison with IS

Gaussians: BDNK vs IS (Hot QCD EOS)

Advantages of BDNK vs IS (Implementation)

- Simpler GR versions of the equations
- Simpler non-linear causality and stability conditions

Equal initial conditions (Recovery)

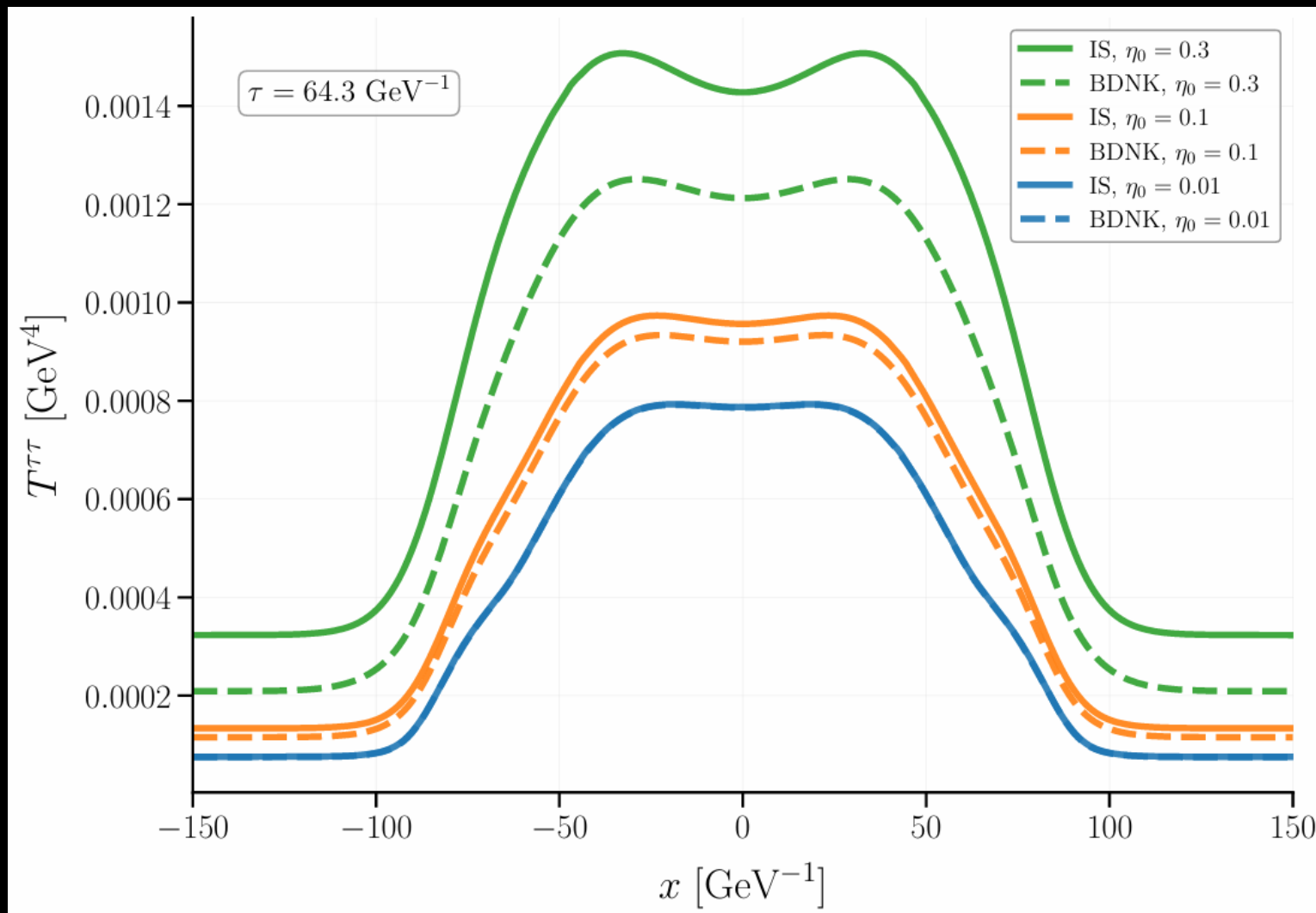
- Straightforward in IS
- Tricky in BDNK
- Approach here:

$$\{\varepsilon(\tau_0), u^\mu(\tau_0); \tau_{BDNK}\}$$

$$T_{BDNK}^{\mu\nu}(\tau_0) = T_{IS}^{\mu\nu}(\tau_0)$$

$$\{\varepsilon(\tau_0), u^\mu(\tau_0), \Pi, \pi^{\mu\nu}; \tau_{IS}\}$$

Minimal IS theory : MUSIC



$$T_{IS}^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$\tau_\Pi u^\mu \partial_\mu \Pi + \Pi = -\zeta \partial_\mu u^\mu$$

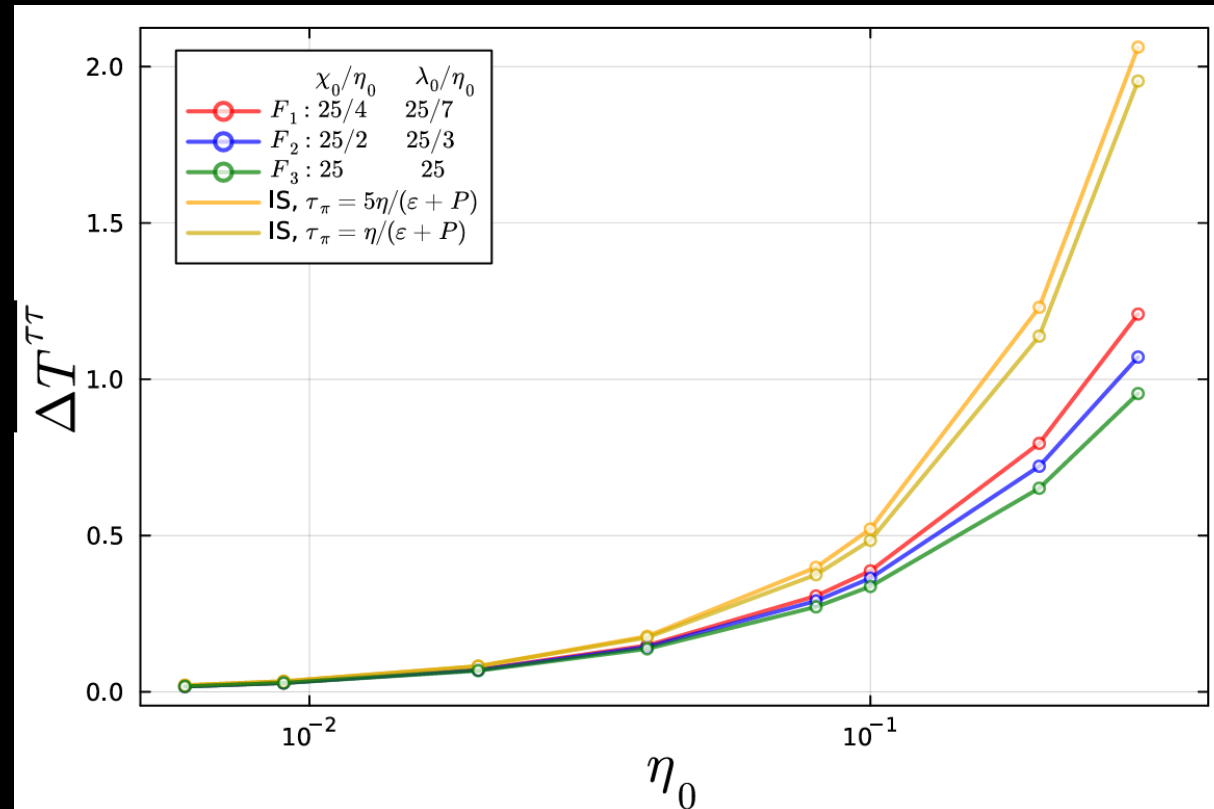
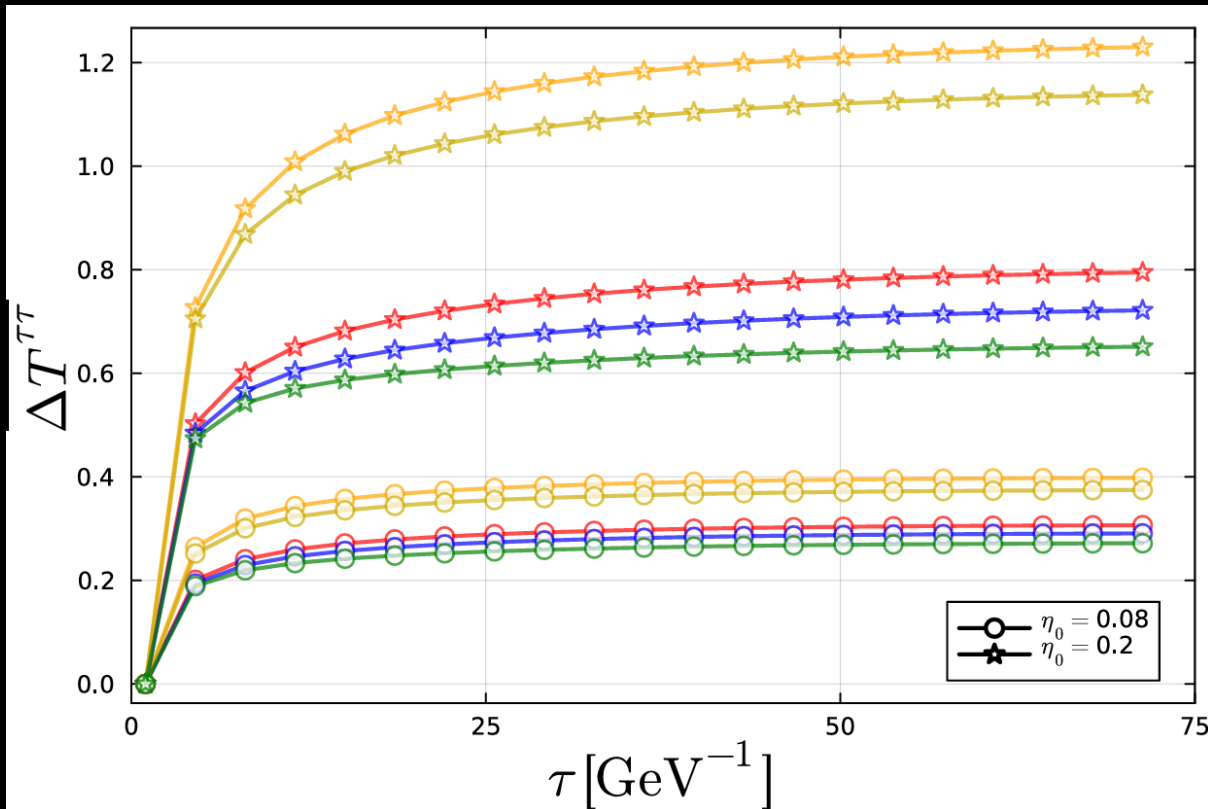
$$\tau_\pi u^\sigma \partial_\sigma (\Delta_{\alpha\beta}^{\mu\nu} \pi^{\alpha\beta}) + \pi^{\mu\nu} = -2\eta \sigma^{\mu\nu}$$

Comparison with IS

- Largest differences coming from initial expansion
- BDNK approaches equilibrium faster
 - Because of finite time relaxation in IS

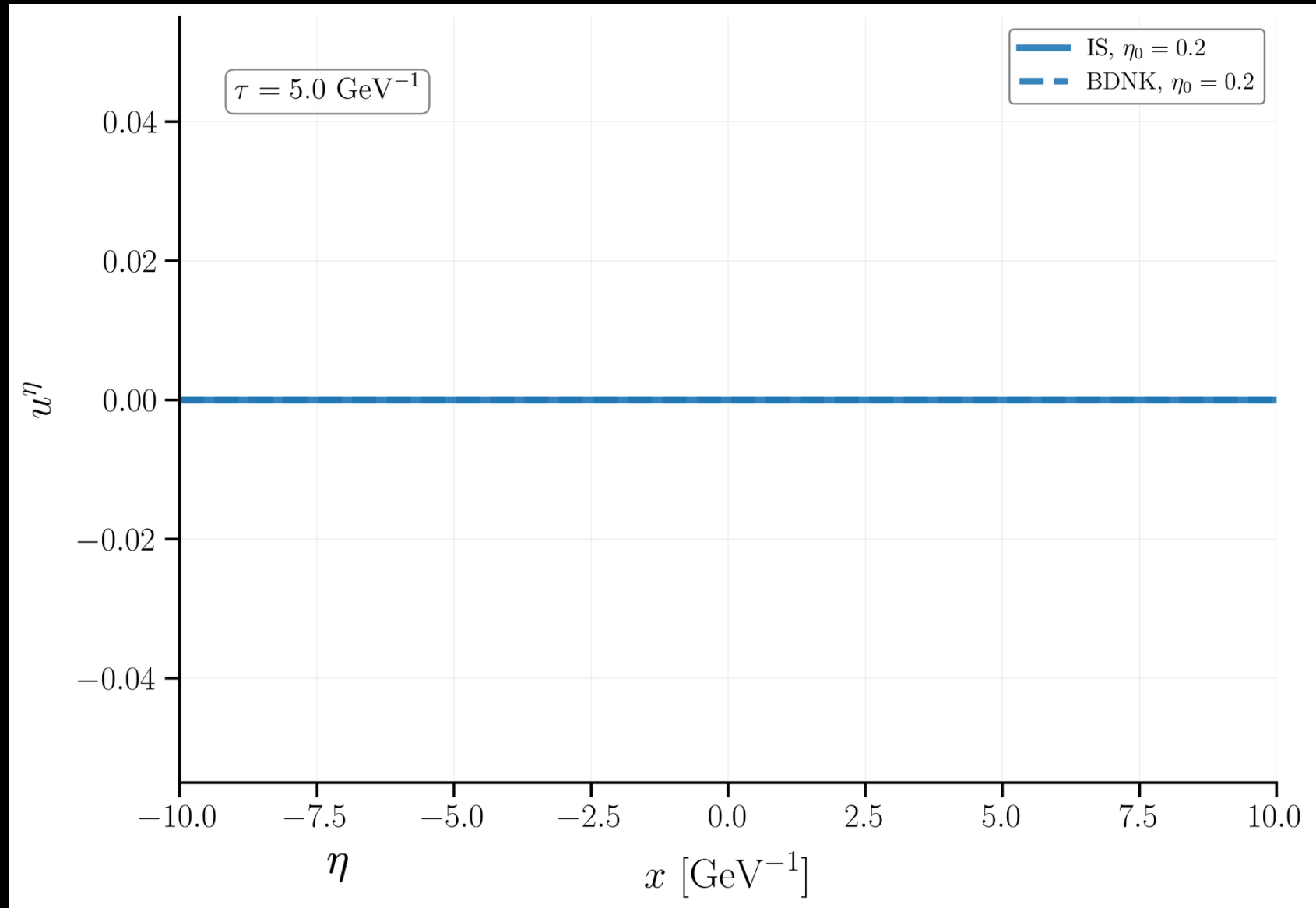
$$\overline{\Delta u}(t_n) := \frac{\sum_i^N |U_i^n - U_{0,i}^n|}{\sum_i^N |U_{0,i}^n|}$$

Normalized absolute difference w.r.t
ideal averaged over the domain

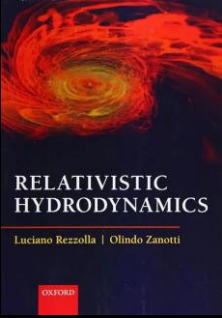


Comparison with IS

- Comparison in the longitudinal direction: Velocity
- MUSIC: Artifacts in near the edges



Causality & Stability Illustration



Eckart heat-flux constitutive relation, schematically

$$a^\mu = u^\nu \nabla_\nu u^\mu$$

$$u_\mu q^\mu = 0$$

$$q^\mu = -\kappa (\Delta^{\mu\nu} \nabla_\nu T + T a^\mu)$$



LRF

$$\mathbf{q} = -\kappa \nabla T.$$



$$c_V \partial_t T - \kappa \nabla^2 T = 0.$$

$$u^\mu = (1, \mathbf{0}).$$

$$\partial_t \epsilon + \nabla \cdot \mathbf{q} = 0.$$



$$\partial_t T^{00} + \partial_i T^{i0} = 0,$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + (P + \Pi) \Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}.$$

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$

IDEAL

Viscous Terms



$$T^{00} = \epsilon,$$

$$T^{i0} = q^i.$$



$$c_V \equiv \left(\frac{\partial \epsilon}{\partial T} \right)_V, \quad \text{Heat Diffusion Equation (Parabolic)}$$

$$G(x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right), \quad t > 0.$$

It corresponds to the initial condition

$$G(x, 0) = \delta(x).$$

Instantaneous response everywhere!

$$T(x, t) = \int_{-\infty}^{\infty} G(x - x', t) T_0(x') dx'$$

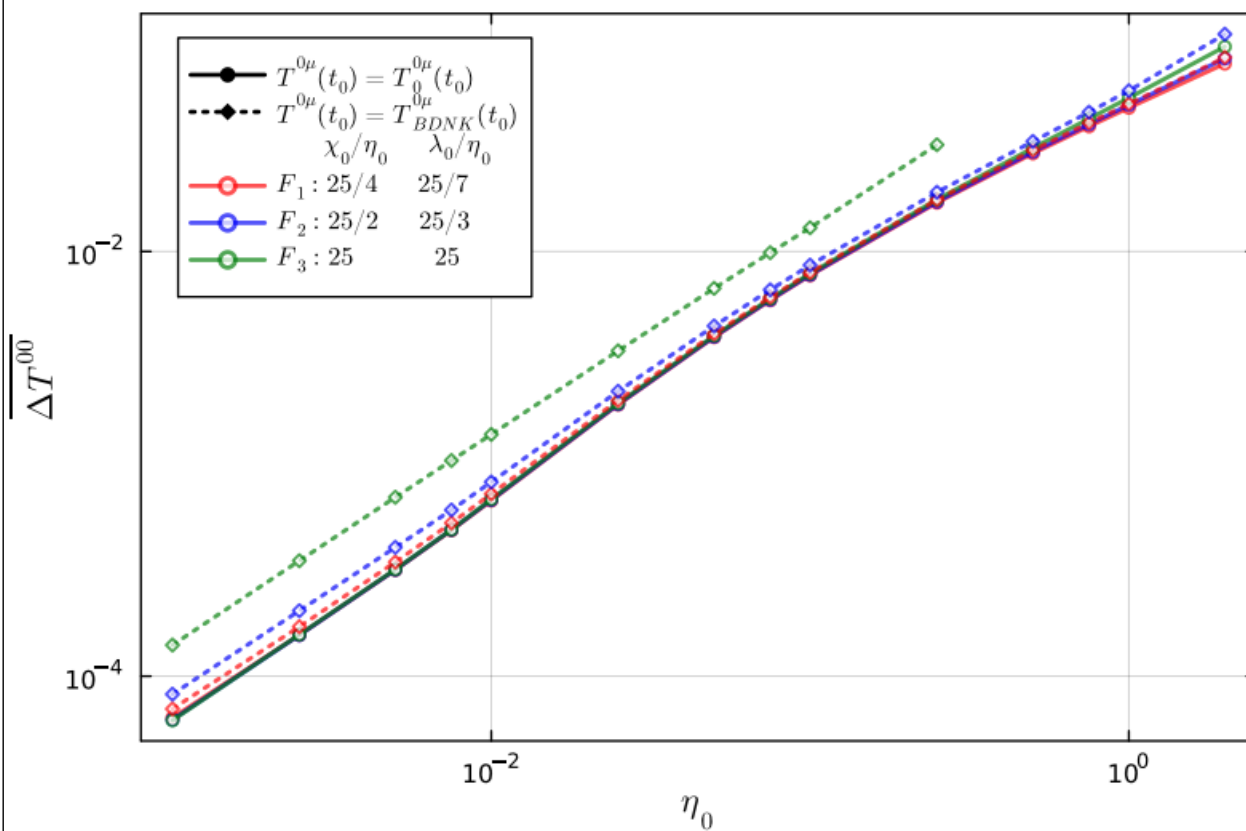
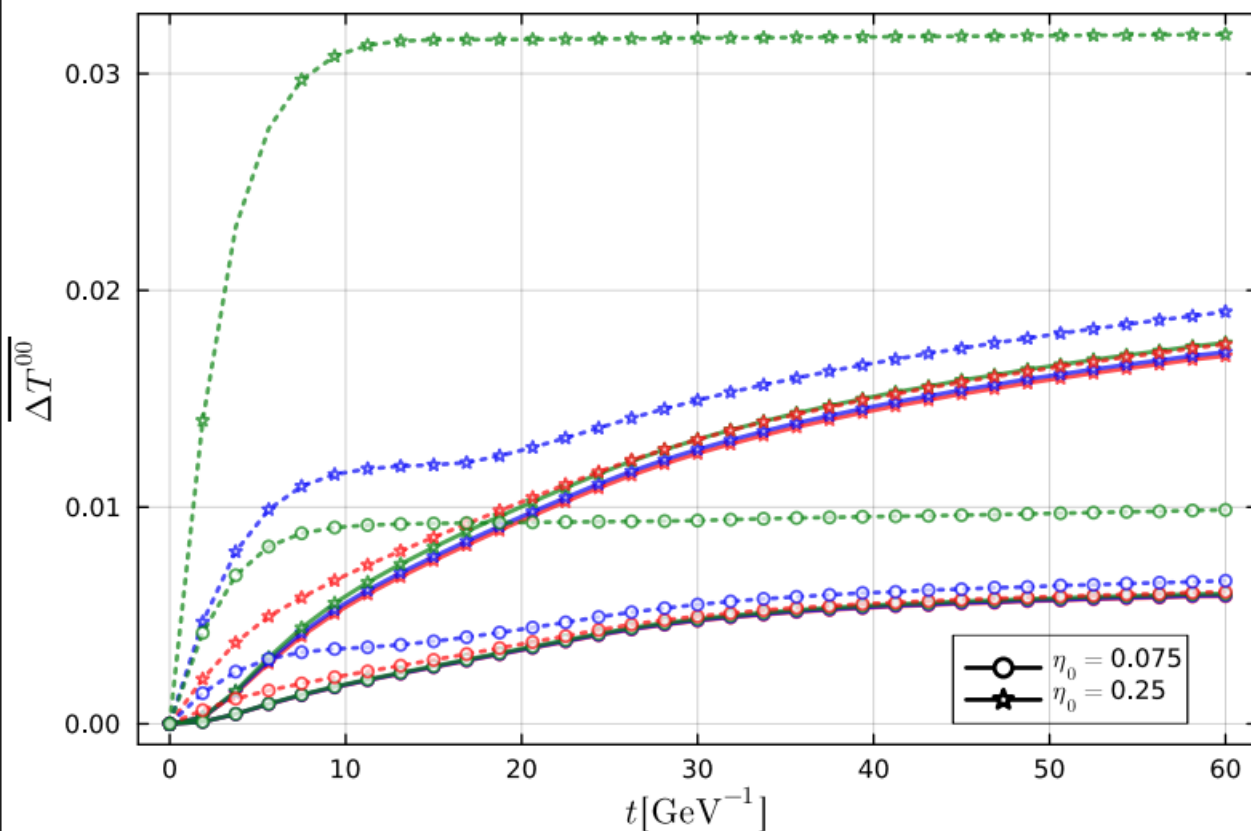
$$T(t, x) \neq 0 \quad \forall x, t > 0$$

Validation III: Shock Tube tests

Quantifying departure from Ideal Hydrodynamics

$$\Delta u(t_n) := \frac{\langle |U_i^n - U_{0,i}^n| \rangle}{\langle |U_{0,i}^n| \rangle}$$

Relative absolute difference
averaged over the grid



χ_0 λ_0

BDNK Frame

(Causality and Stability)

Independence of frame
parameters : Frame Robust

✓ $T^{0\mu}(t_0, \vec{x}) = T_0^{0\mu}(t_0, \vec{x})$

$T^{0\mu}(t_0, \vec{x}; \eta_0, \chi_0, \lambda_0)$ ✗

Viscous Transport Coefficients

η_0 $\zeta = \sigma = 0$

Ideal/Inviscid $\eta_0 \rightarrow 0$

Kelvin Helmholtz Setup

Adopted from Lecoanet et. al. 2015

$$\epsilon = 1$$

$$U^x = u_{\text{flow}} \left[\tanh \left(\frac{y - y_1}{a} \right) - \tanh \left(\frac{y - y_2}{a} \right) - 1 \right]$$

$$U^y = A \sin(2\pi x) \left[\exp \left(- \left[\frac{y - y_1}{\sigma} \right]^2 \right) + \exp \left(- \left[\frac{y - y_2}{\sigma} \right]^2 \right) \right],$$

BDNK EOMs – Constitutive Relations

$$T_{\mu\nu} = (\varepsilon + \mathcal{A}) u_\mu u_\nu + (P + \Pi) \Delta_{\mu\nu} - 2\eta \sigma_{\mu\nu} + u_\mu \mathcal{Q}_\nu + u_\nu \mathcal{Q}_\mu$$

$$\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

$$\Delta^{\mu\nu\alpha\beta} = (\Delta^{\alpha\mu} \Delta^{\beta\nu} + \Delta^{\alpha\nu} \Delta^{\beta\mu}) / 2 - \Delta^{\mu\nu} \Delta^{\alpha\beta} / 3$$

$$\sigma^{\mu\nu} = \Delta^{\mu\nu\alpha\beta} \nabla_\alpha u_\beta$$

$$\mathcal{A} = \tau_\varepsilon [u^\mu \nabla_\mu \varepsilon + (\varepsilon + P) \nabla_\mu u^\mu] = -\tau_\varepsilon \left(u_\alpha \nabla_\beta T_0^{\alpha\beta} \right)$$

$$\Pi = \frac{\tau_P}{\tau_\varepsilon} \mathcal{A} - \zeta \nabla_\mu u^\mu = -\zeta \nabla_\mu u^\mu - \tau_P \left(u_\alpha \nabla_\beta T_0^{\alpha\beta} \right)$$

$$\mathcal{Q}_\nu = \tau_Q (\varepsilon + P) u^\lambda \nabla_\lambda u_\nu + \beta_\varepsilon \Delta_\nu^\lambda \nabla_\lambda \varepsilon + \beta_n \Delta_\nu^\lambda \nabla_\lambda n$$

Transport coefficients – Simplifying choices

$$\tau_\epsilon = \chi V / \rho, \quad \tau_Q = \lambda V / \rho, \quad \tau_P = \xi \tau_\epsilon$$

$$\lambda > \frac{\chi}{\chi(1 - c_s^2)(1 - \xi) - P'_\epsilon} \qquad \sigma = 0$$

$$\chi > P'_\epsilon \max \left[\frac{1}{(1 - c_s^2)(1 - \xi)}, \frac{1}{c_s^2 \xi} \right]$$

$$\xi = P'_\epsilon = c_s^2 = \frac{1}{3}, \qquad \chi = \frac{3}{4} \frac{\chi_0}{\eta_0}, \qquad \lambda = \frac{3}{4} \frac{\lambda_0}{\eta_0},$$

$$\frac{\chi_0}{\eta_0} \geq 4, \qquad \frac{\lambda_0}{\eta_0} \geq \frac{3\chi_0}{\chi_0 - \eta_0}$$

BDNK EOMs – H Tensor

$$T_{\mu\nu} = (\varepsilon + \mathcal{A}) u_\mu u_\nu + (P + \Pi) \Delta_{\mu\nu} - 2\eta \sigma_{\mu\nu} + u_\mu \mathcal{Q}_\nu + u_\nu \mathcal{Q}_\mu$$

$$T^{\mu\nu} = T_0^{\mu\nu} + \frac{1}{\varepsilon} H^{\mu\nu\alpha\beta} \nabla_\alpha (\varepsilon u_\beta) + 2\beta_n u^{(\mu} \Omega^{j\nu)} \partial_j n$$

$$\begin{aligned} H^{\mu\nu\alpha\beta} = & \tau_\varepsilon u^\mu u^\nu [(\varepsilon + P) \Delta^{\alpha\beta} - \varepsilon u^\alpha u^\beta] + \Delta^{\mu\nu} [(\tau_P(\varepsilon + P) - \zeta) \Delta^{\alpha\beta} - \varepsilon \tau_P u^\alpha u^\beta] - 2\eta \Delta^{\mu\nu\alpha\beta} \\ & + \tau_Q(\varepsilon + P) u^\mu u^\alpha \Delta^{\beta\nu} + \tau_Q(\varepsilon + P) u^\nu u^\alpha \Delta^{\mu\beta} - \varepsilon \beta_\varepsilon u^\mu u^\beta \Delta^{\alpha\nu} - \varepsilon \beta_\varepsilon u^\nu u^\beta \Delta^{\mu\alpha} \\ & - \frac{n\beta_n}{u^0} (u^\mu \Delta^{\nu 0} + u^\nu \Delta^{\mu 0}) \Delta^{\alpha\beta} \end{aligned}$$

$$C_i^\mu = \delta_i^\mu - \frac{\delta_0^\mu u_i}{u_0} = \Delta_i^\mu - \frac{\Delta_0^\mu u_i}{u_0}, \quad \left(C_{i\mu} = g_{\alpha\mu} C_i^\alpha = g_{\mu i} - \frac{g_{\mu 0} u_i}{u_0} \right).$$

$$P_{A\mu} = \begin{cases} u_\mu & \text{if } A = \theta \\ C_{i\mu} & \text{if } A = i \end{cases} \quad \text{and} \quad \tilde{P}^{A\mu} = \begin{cases} -u^\mu & \text{if } A = \theta \\ \Delta^{i\mu} & \text{if } A = i \end{cases}.$$

Divergence Cleaning

$$\partial_t \mathbf{B} - \nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

Dedner et al. (2002) : GLM methods to resolve numerical errors causing instability in MHD simulations

$$\partial_t \mathbf{B} - \nabla \times \mathbf{E} + \nabla \phi = 0,$$

$$\partial_t \phi + c_h^2 \nabla \cdot \mathbf{B} = -\frac{\phi}{\tau},$$

- Full first-order formulation introduces **curl constraints**
- Introduce an auxiliary **curl-cleaning (CC) field** to propagate and damp constraint violations
- Cleaning/damping controlled by adjustable coefficients

$$c_h \quad \tau$$

$$\nabla \cdot \mathbf{B} \approx 0$$

First Order Formulism – Curl Cleaning

$$\mathbf{X} = \nabla \psi$$

$$\nabla \times \mathbf{X} = 0$$

New variables which are gradients requires methods to ensure curl is zero

$$\partial_t Y - \nabla \cdot \mathbf{X} = -\psi$$

$$\partial_t \mathbf{X} - \nabla Y = 0$$

$$\partial_t \psi = Y.$$

$$Y = \partial_t \psi,$$

$$\mathbf{X} = \nabla \psi,$$

Ensure Curl does not grow

$$\partial_t \mathbf{X} - \nabla Y + \nabla \times \varphi = 0,$$

$$\partial_t \varphi - c_\varphi^2 \nabla \times \mathbf{X} = -\frac{\varphi}{\tau_1},$$

$$\partial_t^2 (\nabla \times \varphi) - c_\varphi^2 \nabla^2 (\nabla \times \varphi) = -\frac{\partial_t (\nabla \times \varphi)}{\tau_1},$$

$$\partial_t^2 (\nabla \cdot \varphi) = -\frac{\partial_t (\nabla \cdot \varphi)}{\tau_1}.$$

$$Y^A \equiv \partial_t U^A,$$

$$X_i^A \equiv \partial_i U^A, \quad \text{New Variables for flux conservative form}$$

$$Z_i \equiv \partial_i n.$$

BDNK Tensor

$$T^{\mu\nu} = T_0^{\mu\nu} + H^{\mu\nu 0\beta} P_{A\beta} Y^A + H^{\mu\nu j\beta} P_{A\beta} X_j^A + 2\beta_n u^{(\mu} \Omega^{j\nu)} Z_j + H^{\mu\nu\alpha\beta} C_{j\beta} \Gamma_{\alpha\lambda}^j u^\lambda$$

Auxiliary Definitions

$$Y^A = \tilde{P}^{A\lambda} M_{\lambda\mu}^{-1} \left[T^{0\mu} - T_0^{0\mu} - H^{\mu 0 k\beta} P_{B\beta} X_k^B - 2\beta_n u^{(\mu} \Omega^{k0)} Z_k - H^{\mu 0\alpha\beta} C_{k\beta} \Gamma_{\alpha\lambda}^k u^\lambda \right],$$

$$T^{ij} = T_0^{ij} + H^{ij 0\beta} P_{A\beta} Y^A + H^{ijk\beta} P_{A\beta} X_k^A + 2\beta_n u^{(i} \Omega^{kj)} Z_k + H^{ij\alpha\beta} C_{k\beta} \Gamma_{\alpha\lambda}^k u^\lambda.$$

EOMs

$$\partial_t T^{0\mu} + \partial_i T^{i\mu} = -\Gamma_{\nu 0}^\nu T^{0\mu} - \Gamma_{0\nu}^\mu T^{0\nu} - \Gamma_{\nu i}^\nu T^{i\mu} - \Gamma_{i\nu}^\mu T^{i\nu},$$

$$\partial_t X_i^A - \partial_i Y^A = 0,$$

$$\partial_t Z_i - \partial_i S_n = 0,$$

$$\partial_t U^A = Y^A,$$

$$\partial_t n = S_n.$$

$$Y^A \equiv \partial_t U^A,$$

$$X_i^A \equiv \partial_i U^A, \quad \text{New Variables for flux conservative form}$$

$$Z_i \equiv \partial_i n.$$

BDNK Tensor

$$T^{\mu\nu} = T_0^{\mu\nu} + H^{\mu\nu 0\beta} P_{A\beta} Y^A + H^{\mu\nu j\beta} P_{A\beta} X_j^A + 2\beta_n u^{(\mu} \Omega^{j\nu)} Z_j + H^{\mu\nu\alpha\beta} C_{j\beta} \Gamma_{\alpha\lambda}^j u^\lambda$$

Auxiliary Definitions

$$Y^A = \tilde{P}^{A\lambda} M_{\lambda\mu}^{-1} \left[T^{0\mu} - T_0^{0\mu} - H^{\mu 0 k \beta} P_{B\beta} X_k^B - 2\beta_n u^{(\mu} \Omega^{k0)} Z_k - H^{\mu 0 \alpha \beta} C_{k\beta} \Gamma_{\alpha\lambda}^k u^\lambda \right],$$

$$T^{ij} = T_0^{ij} + H^{ij 0 \beta} P_{A\beta} Y^A + H^{ijk\beta} P_{A\beta} X_k^A + 2\beta_n u^{(i} \Omega^{kj)} Z_k + H^{ij\alpha\beta} C_{k\beta} \Gamma_{\alpha\lambda}^k u^\lambda.$$

EOMs

$$\partial_t T^{0\mu} + \partial_i T^{i\mu} = -\Gamma_{\nu 0}^\nu T^{0\mu} - \Gamma_{0\nu}^\mu T^{0\nu} - \Gamma_{\nu i}^\nu T^{i\mu} - \Gamma_{i\nu}^\mu T^{i\nu},$$

$$\partial_t X_i^A - \partial_i Y^A = 0,$$

$$\partial_t Z_i - \partial_i S_n = 0,$$

$$\partial_t U^A = Y^A,$$

$$\partial_t n = S_n.$$

Flux Conservative Form

44 Variables

$$\partial_t \mathcal{Q} + \partial_i F_i = \mathcal{S},$$

$$\mathcal{Q} = \begin{bmatrix} T^{0\mu} \\ X_1^A \\ X_2^A \\ X_3^A \\ Z_1 \\ Z_2 \\ Z_3 \\ U^A \\ n \\ \varphi_1^A \\ \varphi_2^A \\ \varphi_3^A \\ \phi^A \\ \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \psi \end{bmatrix}, \quad F_i = \begin{bmatrix} T^{i\mu} \\ \epsilon_{1ik}\varphi_k^A - \delta_{1i}Y^A \\ \epsilon_{2ik}\varphi_k^A - \delta_{2i}Y^A \\ \epsilon_{3ik}\varphi_k^A - \delta_{3i}Y^A \\ \epsilon_{1ik}\Psi_k - \delta_{1i}S_n \\ \epsilon_{2ik}\Psi_k - \delta_{2i}S_n \\ \epsilon_{3ik}\Psi_k - \delta_{3i}S_n \\ 0_{4 \times 1} \\ 0 \\ -\kappa_1^A \epsilon_{1ik}X_k^A + \delta_{1i}\phi^A \\ -\kappa_1^A \epsilon_{2ik}X_k^A + \delta_{2i}\phi^A \\ -\kappa_1^A \epsilon_{3ik}X_k^A + \delta_{3i}\phi^A \\ \kappa_2^A \varphi_i^A \\ -\lambda_1 \epsilon_{1ik}Z_k + \delta_{1i}\psi \\ -\lambda_1 \epsilon_{2ik}Z_k + \delta_{2i}\psi \\ -\lambda_1 \epsilon_{3ik}Z_k + \delta_{3i}\psi \\ \lambda_2 \Psi_i \end{bmatrix}, \quad \text{and} \quad \mathcal{S} = \begin{bmatrix} S^\mu \\ 0_{4 \times 1} \\ 0_{4 \times 1} \\ 0_{4 \times 1} \\ 0 \\ 0 \\ 0 \\ Y^A \\ S_n \\ -\beta_1^A \varphi_1^A \\ -\beta_1^A \varphi_2^A \\ -\beta_1^A \varphi_3^A \\ -\beta_2^A \phi^A \\ -\sigma_1 \Psi_1 \\ -\sigma_1 \Psi_2 \\ -\sigma_1 \Psi_3 \\ -\sigma_2 \psi \end{bmatrix}$$

Symbolic Expressions

BDNK Tensor

$$T^{\mu\nu} = T_0^{\mu\nu} + \frac{1}{\varepsilon} H^{\mu\nu\alpha\beta} \nabla_\alpha (\varepsilon u_\beta) + 2\beta_n u^{(\mu} \Omega^{j\nu)} \partial_j n$$

$$T^{\mu\nu} = T_0^{\mu\nu} + H^{\mu\nu 0\beta} P_{A\beta} Y^A + H^{\mu\nu j\beta} P_{A\beta} X_j^A + 2\beta_n u^{(\mu} \Omega^{j\nu)} Z_j + H^{\mu\nu\alpha\beta} C_{j\beta} \Gamma_{\alpha\lambda}^j u^\lambda$$

New Variables

$$Y^A \equiv \partial_t U^A,$$

$$X_i^A \equiv \partial_i U^A,$$

$$Z_i \equiv \partial_i n.$$

EOMs

$$\begin{aligned} \partial_t T^{0\mu} + \partial_i T^{i\mu} &= -\Gamma_{\nu 0}^\nu T^{0\mu} - \Gamma_{0\nu}^\mu T^{0\nu} - \Gamma_{\nu i}^\nu T^{i\mu} - \Gamma_{i\nu}^\mu T^{i\nu}, \\ \partial_t X_i^A - \partial_i Y^A &= 0, \\ \partial_t Z_i - \partial_i S_n &= 0, \\ \partial_t U^A &= Y^A, \\ \partial_t n &= S_n. \end{aligned}$$

Auxiliary Definitions

$$Y^A = \tilde{P}^{A\lambda} M_{\lambda\mu}^{-1} \left[T^{0\mu} - T_0^{0\mu} - H^{\mu 0 k \beta} P_{B\beta} X_k^B - 2\beta_n u^{(\mu} \Omega^{k 0)} Z_k - H^{\mu 0 \alpha \beta} C_{k\beta} \Gamma_{\alpha\lambda}^k u^\lambda \right],$$

$$T^{ij} = T_0^{ij} + H^{ij 0 \beta} P_{A\beta} Y^A + H^{ij k \beta} P_{A\beta} X_k^A + 2\beta_n u^{(i} \Omega^{kj)} Z_k + H^{ij \alpha \beta} C_{k\beta} \Gamma_{\alpha\lambda}^k u^\lambda.$$

Symbolic Expressions for Code

BDNK Tensor

$$T^{\mu\nu} = T_0^{\mu\nu} + H^{\mu\nu 0\beta} P_{A\beta} Y^A + H^{\mu\nu j\beta} P_{A\beta} X_j^A + 2\beta_n u^{(\mu} \Omega^{j\nu)} Z_j + H^{\mu\nu\alpha\beta} C_{j\beta} \Gamma_{\alpha\lambda}^j u^\lambda$$

$$Y^A = \tilde{P}^{A\lambda} M_{\lambda\mu}^{-1} \left[T^{0\mu} - T_0^{0\mu} - H^{\mu 0 k \beta} P_{B\beta} X_k^B - 2\beta_n u^{(\mu} \Omega^{k 0)} Z_k - H^{\mu 0 \alpha \beta} C_{k\beta} \Gamma_{\alpha\lambda}^k u^\lambda \right], \quad \sim 10^5$$

$$T^{ij} = T_0^{ij} + H^{ij 0 \beta} P_{A\beta} Y^A + H^{ij k \beta} P_{A\beta} X_k^A + 2\beta_n u^{(i} \Omega^{kj)} Z_k + H^{ij \alpha \beta} C_{k\beta} \Gamma_{\alpha\lambda}^k u^\lambda. \quad \sim 10^3$$

- Large number of terms via contractions.
- xAct symbolic simplifications reduces this significantly

Simulating BDNK

- BDNK EOMs are second order
- Finite Volume Method
 - Equations cast in a First Order Flux Conservative Form.
 - Volume is divided into cells.
 - Discretized variables are cell-averaged and derivatives have to be computed on interfaces.

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$

$$\nabla_\mu T^{\mu\nu} = 0$$

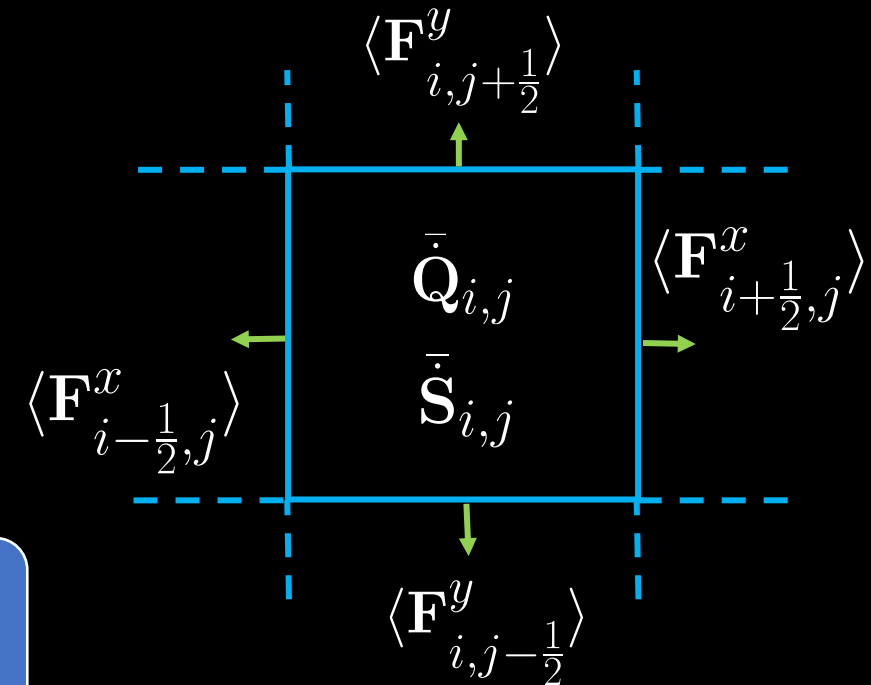
$$\partial_t \mathbf{Q} + \partial_i \mathbf{F}_i = \mathbf{S}$$

Variable

Flux

Sources

$$\bar{\dot{\mathbf{Q}}}_{i,j} + \frac{\langle \mathbf{F}_{i+\frac{1}{2},j}^x \rangle - \langle \mathbf{F}_{i-\frac{1}{2},j}^x \rangle}{\Delta x} + \dots \text{"y term"} \dots = \bar{\mathbf{S}}_{i,j}$$



There is a plethora of numerical techniques to compute the derivatives on cell interfaces

Simulating BDNK

$$\partial_t \mathbf{Q} + \partial_i \mathbf{F}_i = \mathbf{S}$$

Variable

Flux

Sources

First Order

New variables lead to new constraint equations

Numerically, they are violated

Need additional auxiliary variables and extra EOMs

$$Y^1 \equiv \partial_t U^1$$

$$X_i^1 \equiv \partial_i U^1$$

$$\partial_t X_i^A - \partial_i Y^A = 0$$

$$\partial_t X_i^1 + \epsilon_{ijk} \partial_j \varphi_k^1 - \partial_i Y^1 = 0$$

$$\partial_t \varphi_i^1 - \kappa \epsilon_{ijk} \partial_j X_k^1 + \partial_i \phi^1 = -\beta \varphi_i^1$$

$$\{T^{0\mu}, X_i^A, \log \epsilon, U^i, n, \varphi_i^A, \phi^A, \Psi_i, \psi\}$$

Hybrid

Derivatives are numerically evaluated

Not exactly flux-conservative

Fewer variables : Computationally cheaper

$$\{T^{0\mu}, \log \epsilon, U^i, n\}$$

Relativistic Hydrodynamics

Euler Equations

In relativity, u^μ
we have 4-velocity
 $\mu = \{0, 1, 2, 3\}$

$\{t, x, y, z\}$

Normalization ($c = 1$)

$$u^\mu u_\mu = -1$$

Conservation of energy-momentum

$$\dot{\vec{u}} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla P \quad \nabla_\mu T^{\mu\nu} = 0$$

Conservation of mass becomes conservation of 4-current

$$\dot{\rho} + \vec{u} \cdot \nabla \rho = 0 \quad \nabla_\mu J^\mu = 0$$

Ideal Fluid Energy-Momentum Tensor

$$T_0^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$

4-Current

$$J^\mu = \rho u^\mu$$

Rest Frame

$$u^\mu = (1, 0, 0, 0)$$

$$J^\mu = (\rho, 0, 0, 0)$$

$$T^{\mu\nu} \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}$$

Metric Tensor

$$g^{\mu\nu}$$

Einstein's equations : Energy-Momentum $\leftrightarrow$ Spacetime Geometry

$$\begin{pmatrix} e & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Shock smoothing

