CR Propagation Code Project with MHD-Consistent Transport Coefficient Modeling

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Collaborate with

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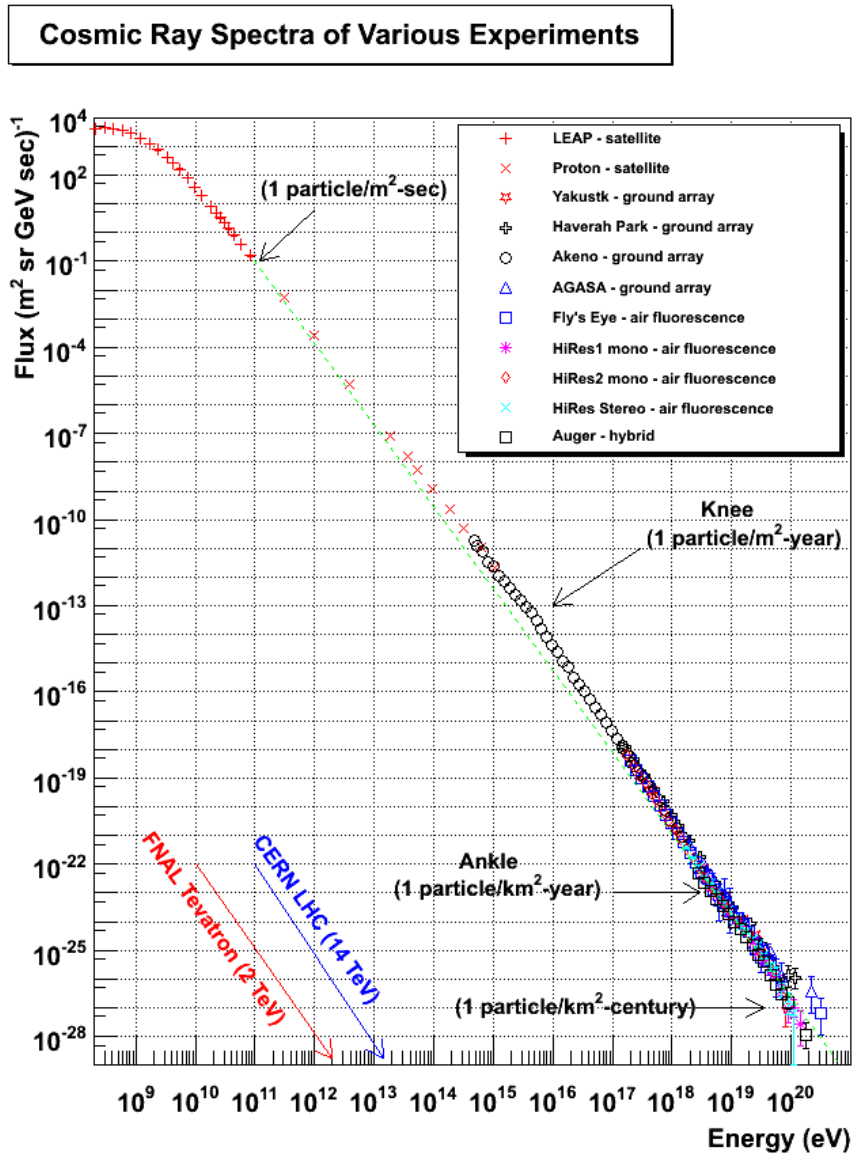
Cosmic-ray

- Cosmic-ray (CR)
  - High-energy charged particles from space
  - Non-thermal energy spectrum:

    Almost single power-law (not Maxwellian)
  - Composed primarily of protons, but also includes heavier nuclei and leptons
- Central problem in cosmic-ray physics
  - **“Where do cosmic-rays come from?”**
  - CRs reach the earth with their direction randomly changed by the magnetic field
  - The direction of arrival and the direction of the source are completely different

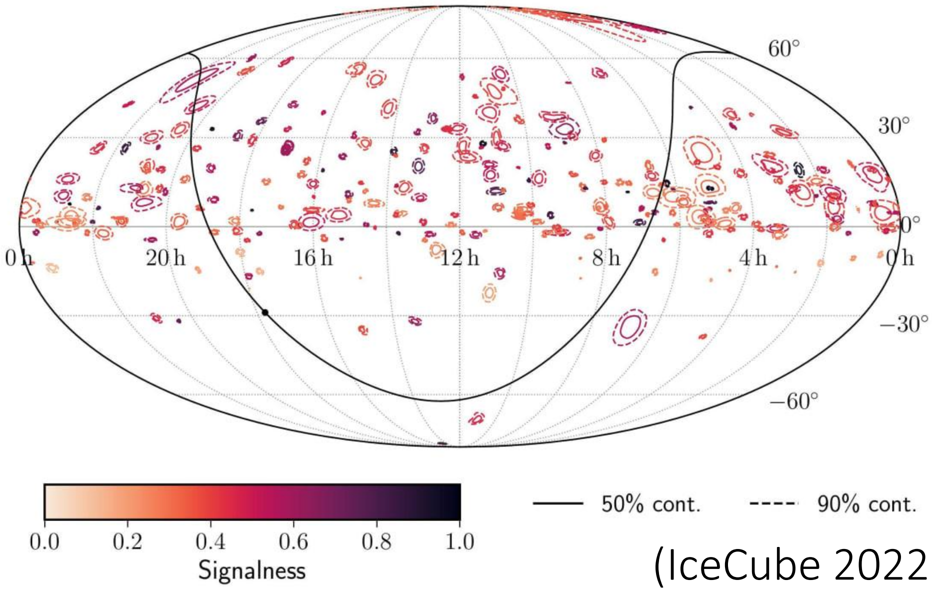


(by William F. Hanlon)

⇒ Neutral particles as secondary products (γ, ν) can be a clue!

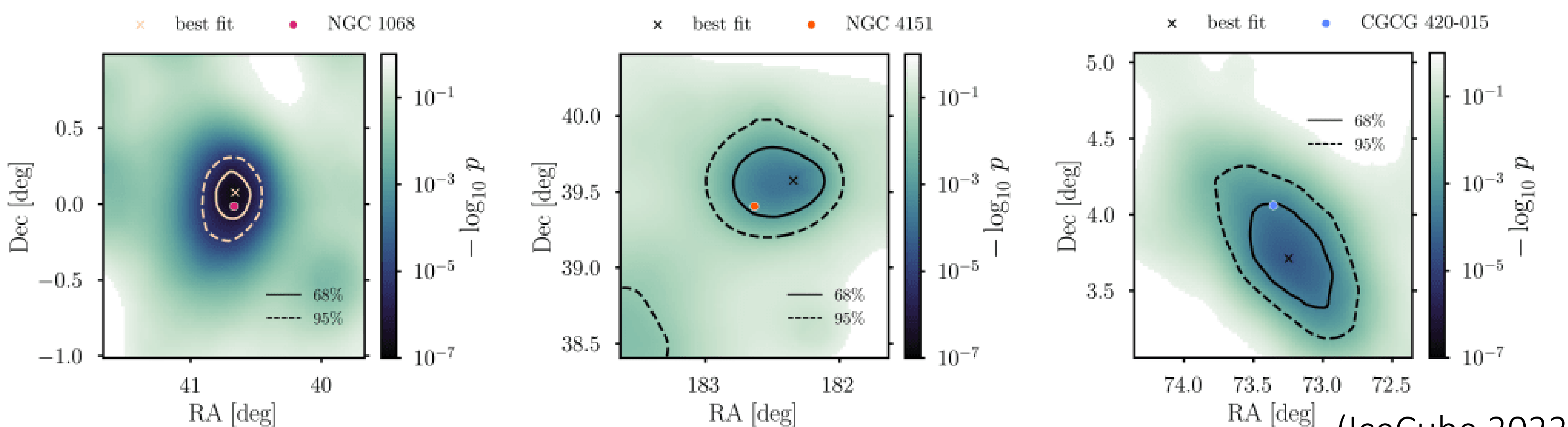
Astrophysical Neutrinos

- Astrophysical neutrinos
  - IceCube has been detected astrophysical neutrinos
  - Signs of neutrino signals from several Seyfert galaxies have been confirmed!
  - Active galaxies must be a factory of cosmic-ray!



(IceCube 2022)

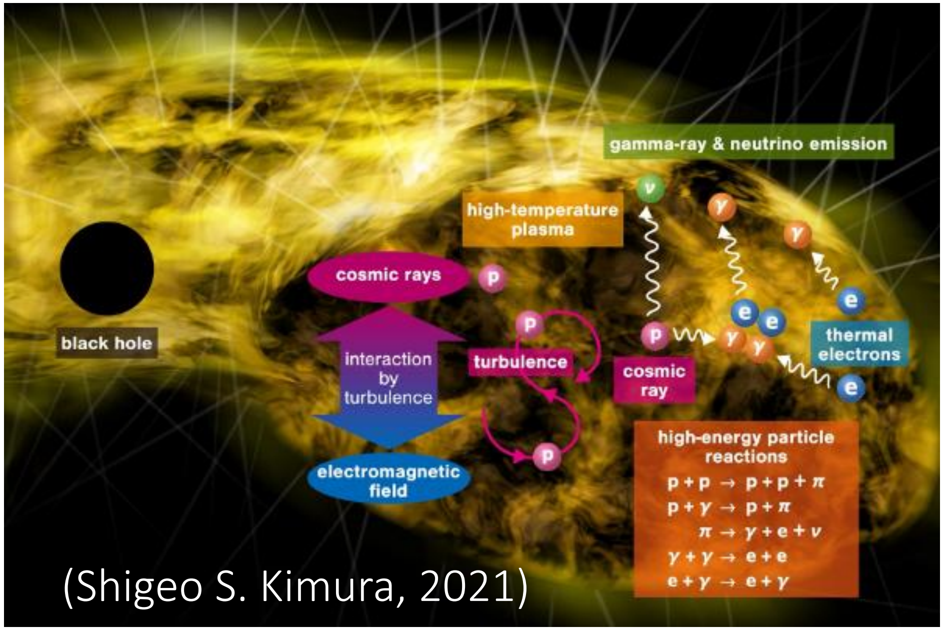
→Astrophysical (1.0)



(IceCube 2022)

Cosmic-ray in active galaxies

- Promising source?
  - Should be not very bright in gamma-rays but make a lot of cosmic rays
  - Accretion disk with collision-less plasma and magnetic turbulence around an AGN
  - Particle acceleration and associated neutrino radiation via photo-meson processes



(Shigeo S. Kimura, 2021)

<u>Key points (one example):</u>

- Generation of pre-accelerated particles via magnetic reconnection in turbulence
- Acceleration and diffusion of cosmic-rays through interaction with turbulence
- Neutrino production via photomeson interactions with the photons from the disk
- Release of cosmic-rays into space through escape from the disk

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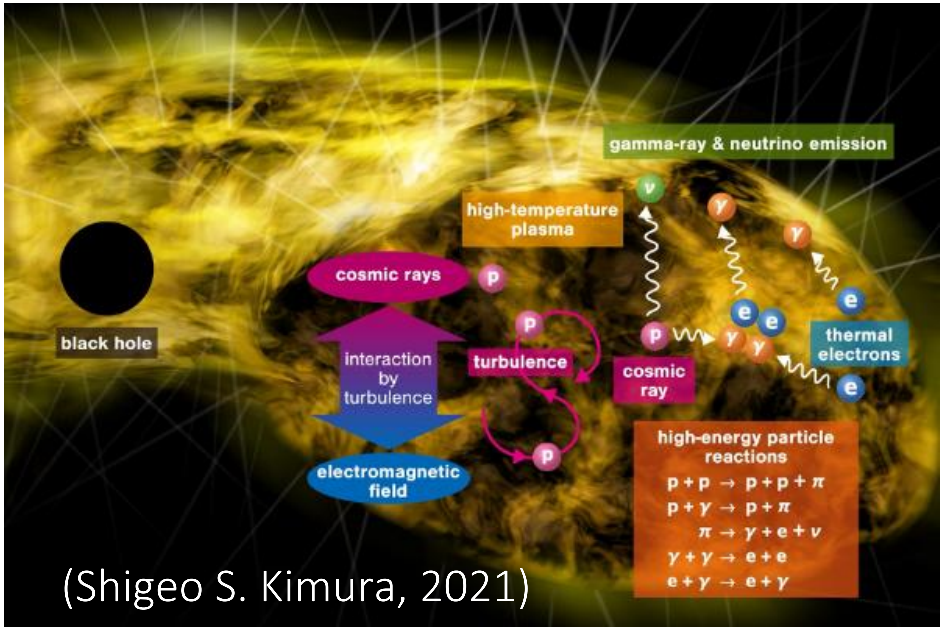

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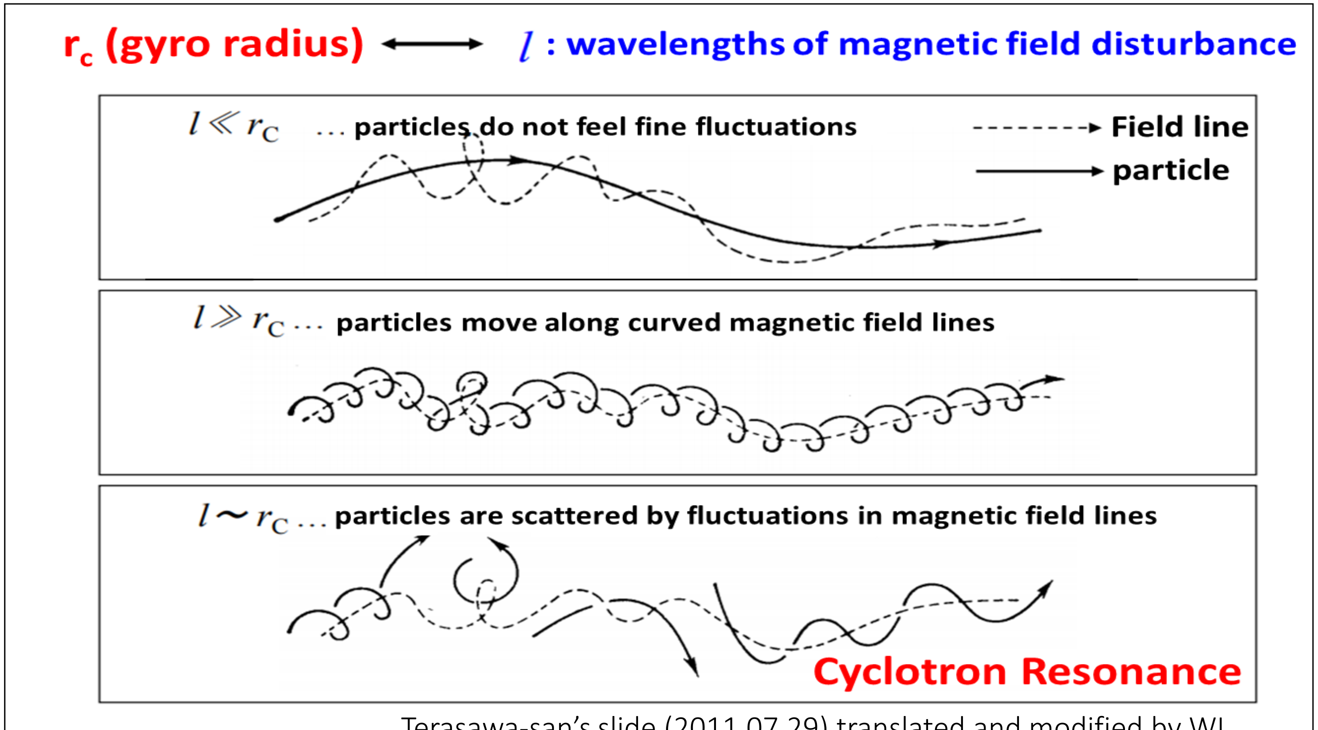
Cosmic-ray acceleration

- The fundamental process of cosmic-ray acceleration = “wave-particle interaction”
  - Cosmic-rays gain energy through interactions with turbulence in collision-less plasma
- The typical scale is determined by the particle's gyro radius
  - Lower-energy particles interact with smaller-scale waves



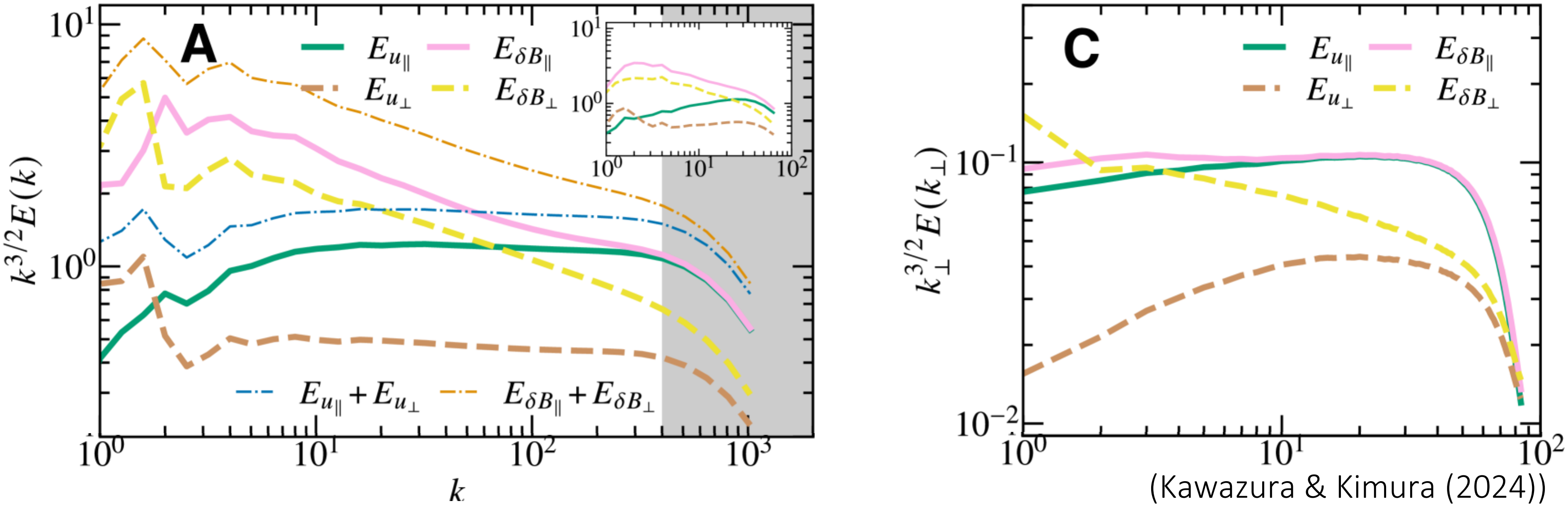
Terasawa-san’s slide (2011.07.29) translated and modified by WI

$$r_L = \frac{E}{eB_0}$$:Gyro radius

Why consider an accretion disk?

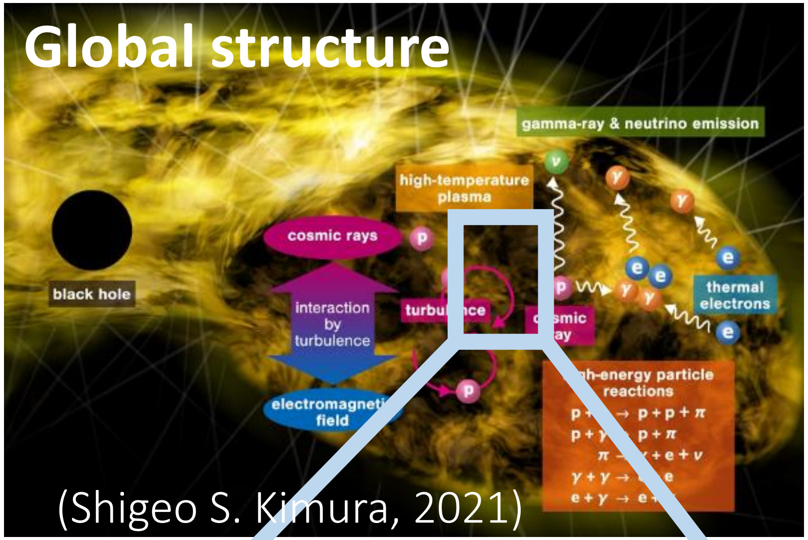
- Spectrum of Magneto-Rotational Instability (MRI) turbulence
  - MRI in accretion disks has a broad injection scale → inertial range has not resolved
  - Kawazura & Kimura (2024): First time ever to resolve from MHD scale to inertial range
  - At much smaller scales, properties are revealed by reduced MHD (Kawazura et al. 2022)
  - Ready to model turbulence from dissipation to MHD scales with a consistent theory!

Aim to solve "acceleration from supra-thermal to ultra-high energy cosmic rays" in accretion disks!



(Kawazura & Kimura (2024))

Ambition



(Shigeo S. Kimura, 2021)

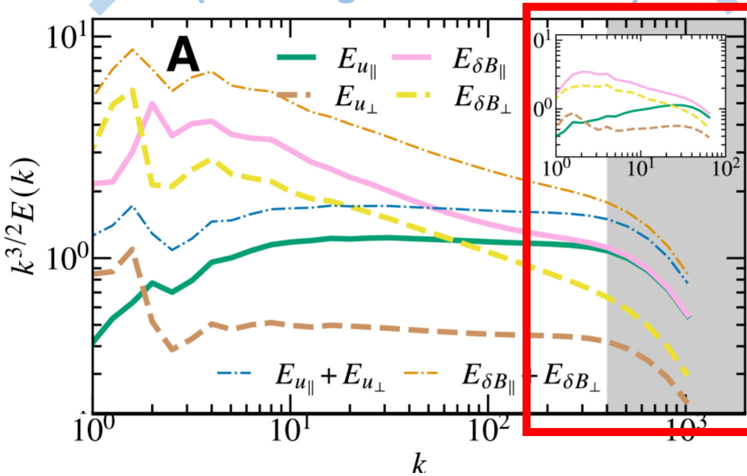
Large scale
(Global disk structure)

High energy CRs
(→ escapable CRs)

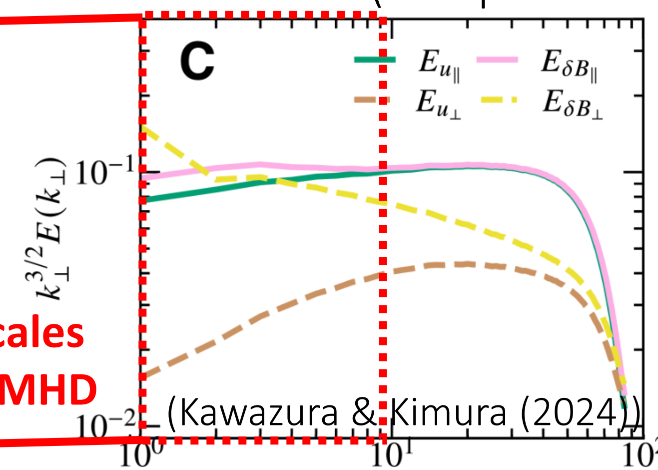
Lower energy CRs
(supra-thermal)

Small scale
(dissipation scale)

Extract a small area
(shearing box simulation)



Very small scales
→ Reduced MHD



(Kawazura & Kimura (2024))

Connecting each scales to achieve a consistent calculation of CR acceleration!

Ambition

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(Global disk structure)

High energy CRs
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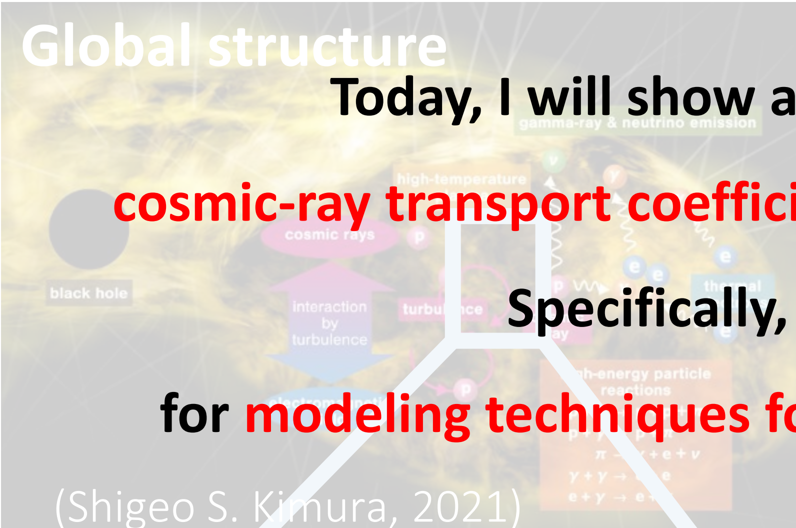


Today, I will show a new methods for modeling

cosmic-ray transport coefficients based on MHD simulation data.

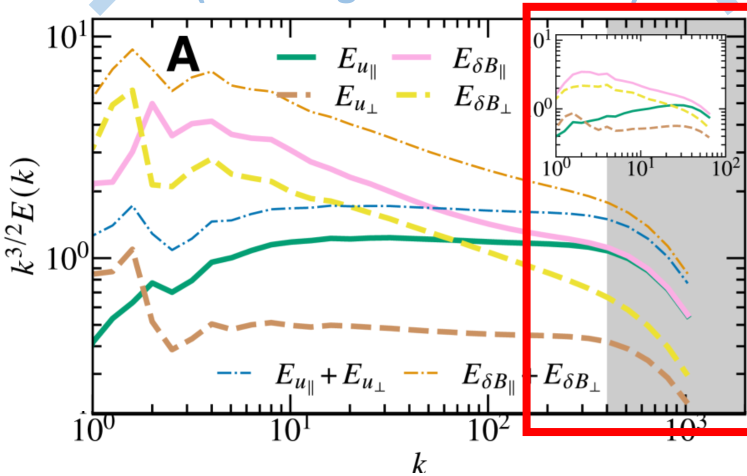
Specifically, I will present results

for modeling techniques for small-scale (subgrid) turbulence.

Lower energy CRs
(supra-thermal)

Small scale
(dissipation scale)

Extract a small area
(shearing box simulation)



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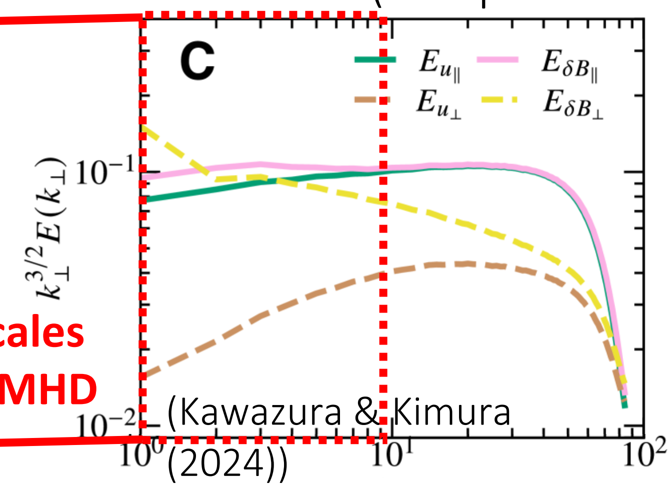
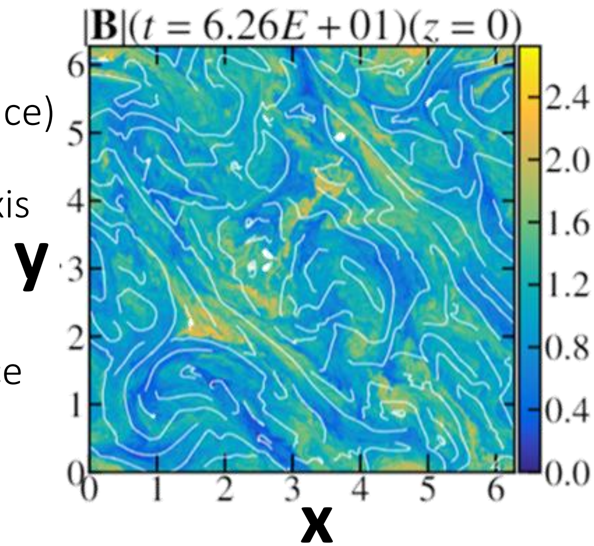
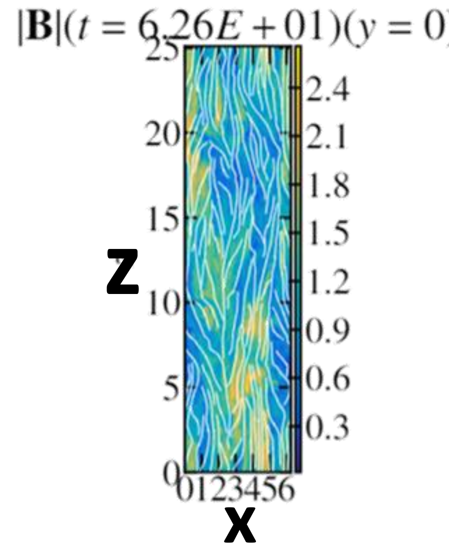


Connecting each scales to achieve a consistent calculation of CR acceleration!

Test Calculation: MHD turbulence

- MHD turbulence (Alfvenic turbulence)
  - Global magnetic field along z-axis
  - Calculate spatial diffusion with MHD data of artificial turbulence

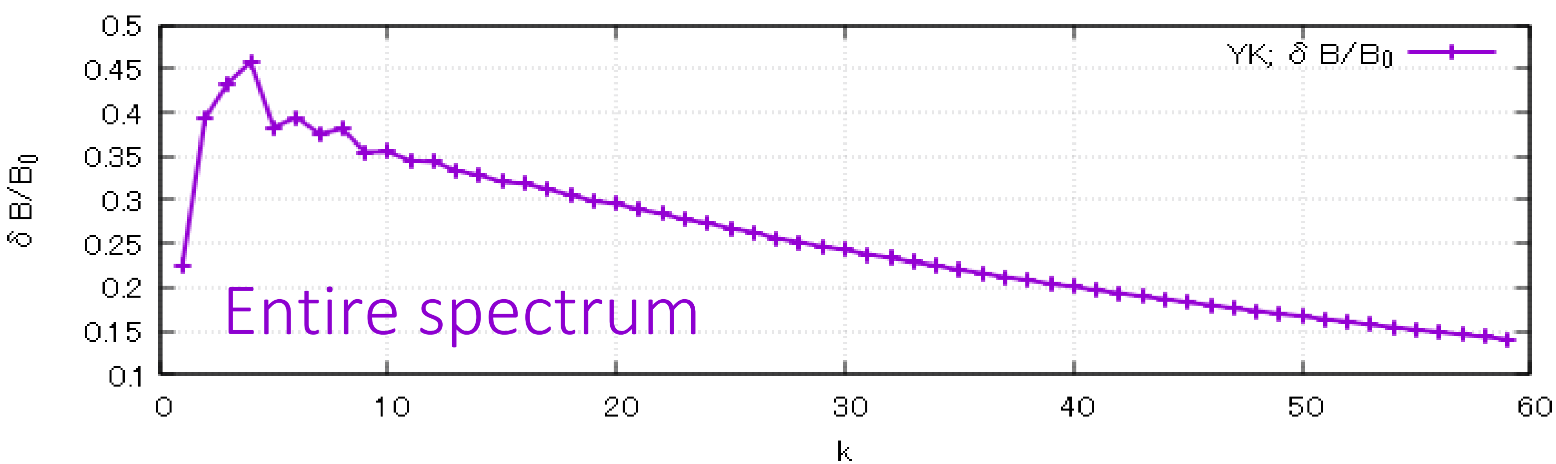




- Model
  - QLT + Alfvenic turbulence w/ magnetostatic (e.g., Blandford & Eichler (1987))

$$D_{\parallel} = \frac{1}{3} r_{\rm L} c \left(\frac{\delta B}{B_0}\right)^{-2}, \qquad D_{\perp} = D_{\parallel} \left(\frac{\delta B}{B_0}\right)^{4}$$

$$r_{\rm L} = \frac{E}{eB_0}$$:Gyro radius



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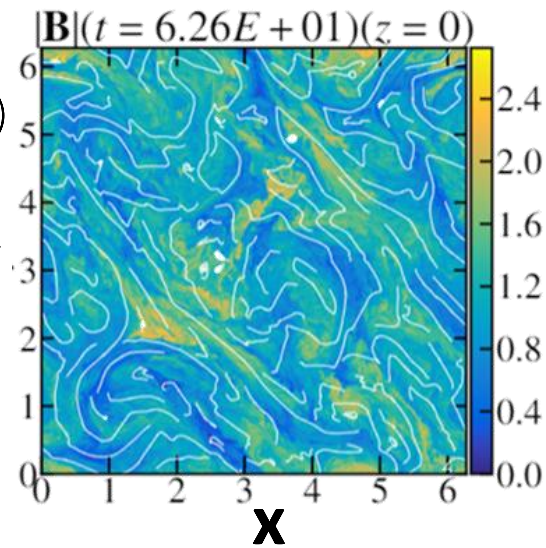

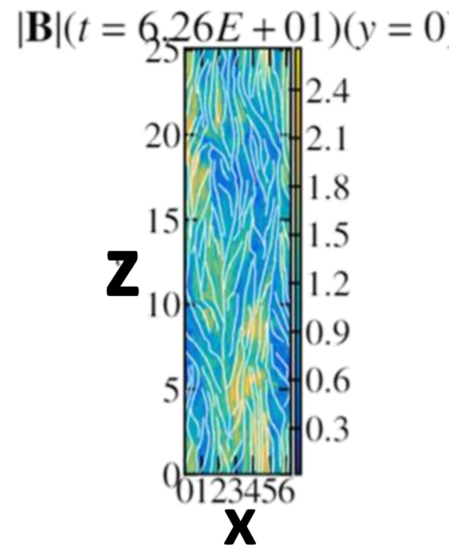

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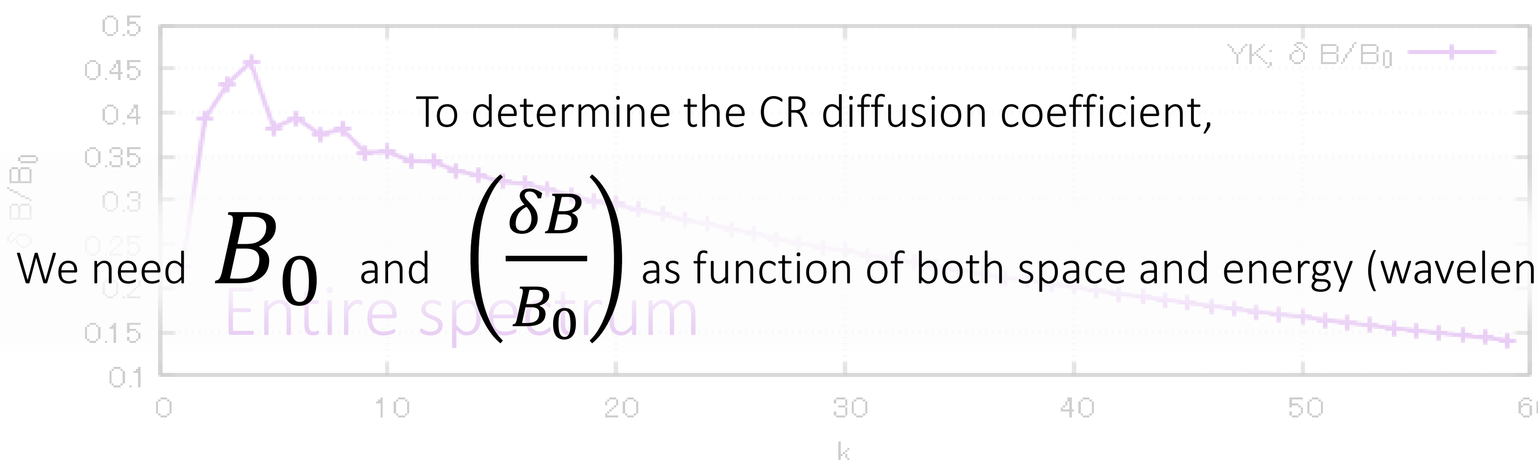
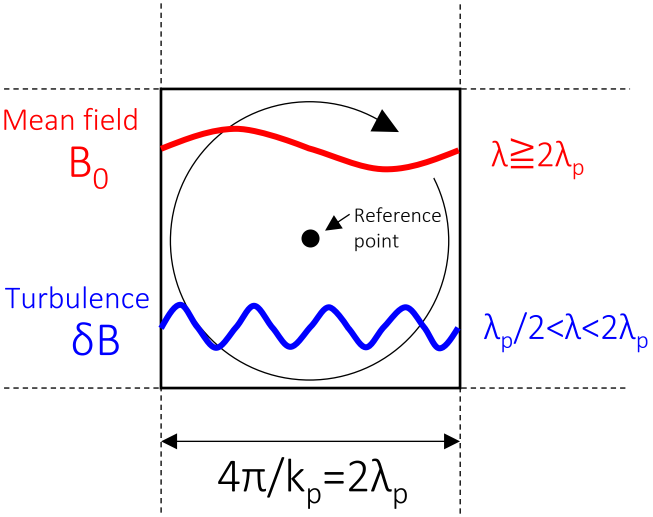


To determine the CR diffusion coefficient,

We need B_0 and $\left(\frac{\delta B}{B_0}\right)$ as function of both space and energy (wavelength)

How do we calculate $\delta B/B_0$ at each point & wave number?

- MHD simulation and turbulence
  - MHD simulation only provides the sum of the "mean field (B_0)" and "turbulent field (δB)".
  - For a given length scale, we need to divide the magnetic field into long-wavelength and short-wavelength components.
- Calculation method
  - First, specify a wave number $k_p=2\pi/\lambda_p$ of the turbulence
  - Take a patch of size $2\lambda_p$ around a reference point
  - Fourier transform of MHD simulation data for each patch
  - The average field and turbulent field are separated
  - The turbulence degree is evaluated from the power ratio
  - Calculate this for each spatial point and the wave number



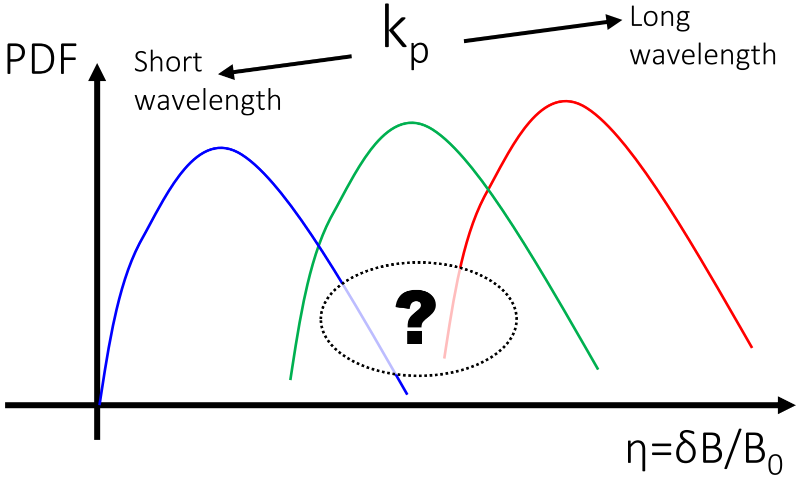
Then we can obtain $\delta B/B_0$ as a function of space and wave number!

Sub-grid modeling

Gyro radius

$$r_L = \frac{E}{eB_0} \propto E$$

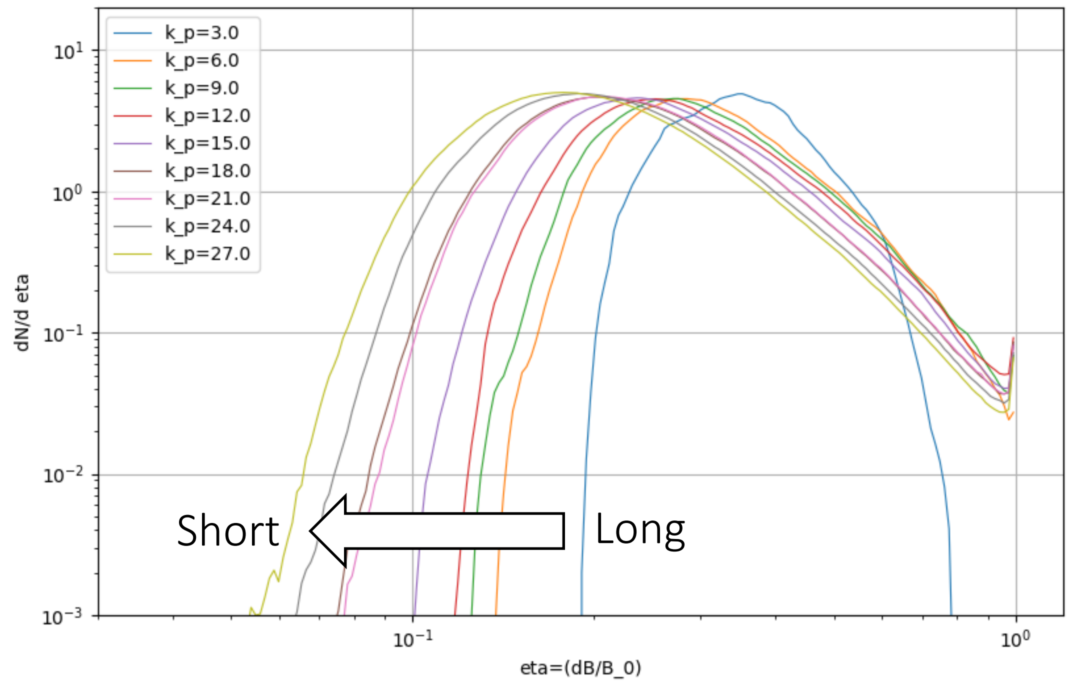
- Wavenumber of turbulence and particle energy
  - Charged particles interact with turbulence of wavelengths around their gyro radius
  - Information on short-wavelength turbulence is necessary to solve the acceleration and propagation of low-energy charged particles
  - However, there are limits to the resolution of MHD calculations, and short-wavelength information must be modeled
- Sub-grid modeling
  - Modeled through the probability distribution function of turbulence
  - The mean field can be roughly obtained from MHD simulation

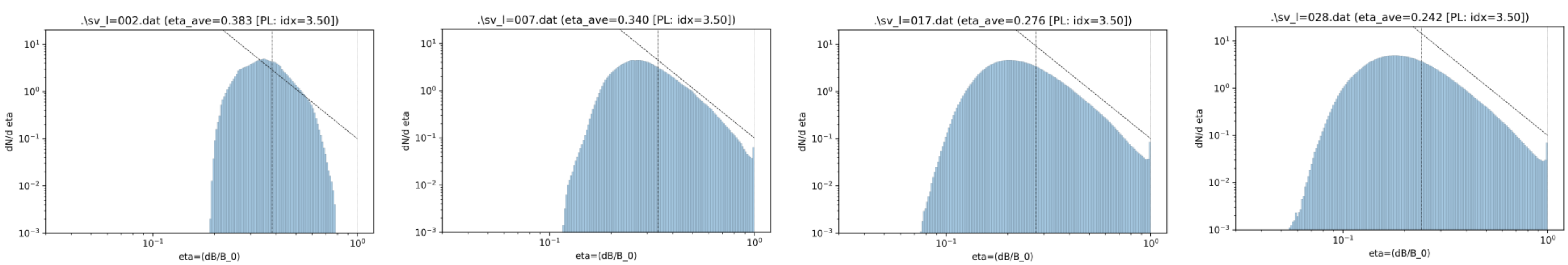


Is it possible to model probability distribution functions of ($\delta B/B_0$)?

Probability Distribution Function (PDF) of η=δB/B_0

- PDF of η=δB/B_0
  - Analysis of box simulation
  - The PDF appears to converge as the wavelength becomes shorter
  - The peak of PDFs follows the average value for the entire box
  - This universality may be due to the intermittency of turbulence



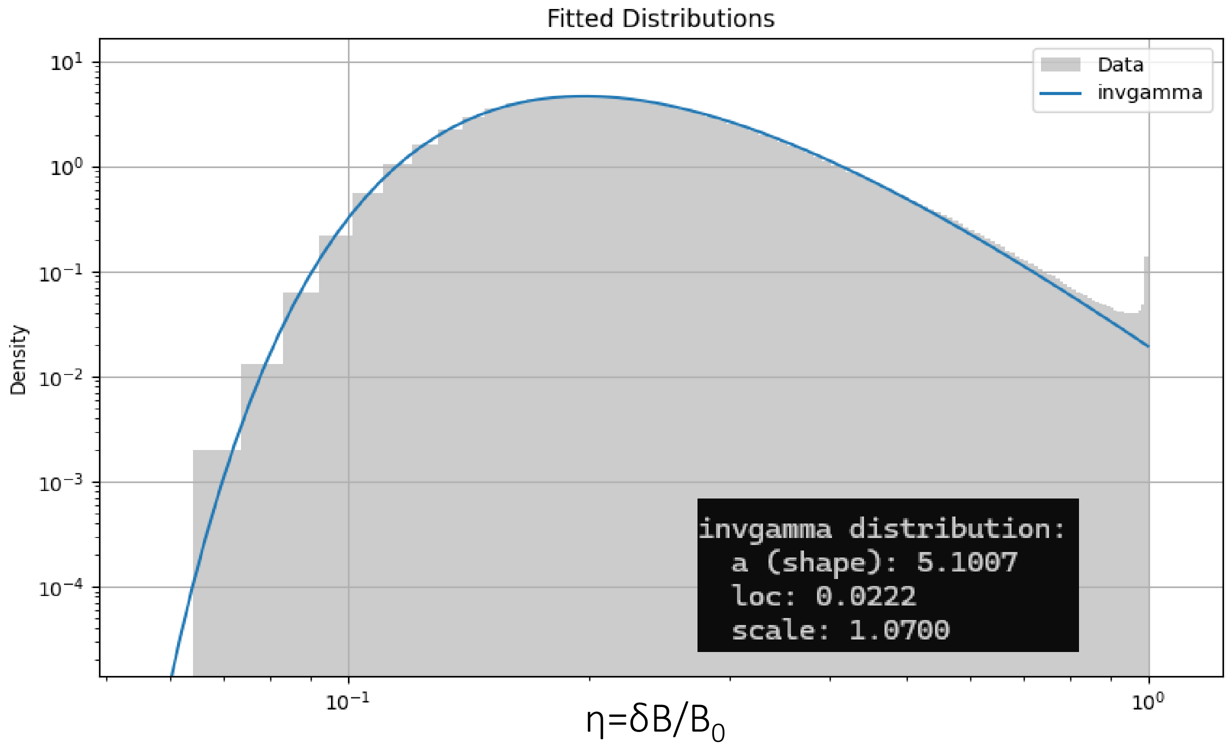


Long wavelength ⟷ Short wavelength

Fitting function

- PDFs are well fitted by the "inverse gamma distribution"
  - α is universal for sufficiently short wavelength (large k_p)
  - β can be obtained from the average value of the box
  - (Even below resolution) predictable from statistical theory of turbulence (e.g. Kolmogorov law)

$$\boxed{f(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{e^{-\beta/x}}{x^{\alpha+1}}}$$



We can use this functional form to model the (much) short wavelength!

Summary / Next steps

- Summary
  - We are developing a code to solve for the acceleration and propagation of cosmic rays consistent with MHD simulations
  - Using a stochastic differential equation approach, a easily extendable and parallelization-efficient code can be designed
  - We have calculated for the spatial diffusion of particles on a box simulation of MHD turbulence
    - We adopted the quasi-linear theory (QLT) and solved the Fokker-Planck equation consistently with MHD simulation
    - We confirm that the code can capture the combined effect of QLT and MHD: field-line wandering
- Next step
  - Calculations for multi-energy case (since this work was a mono energy calculation)
  - Calculations including momentum space diffusion (acceleration)
  - Implementation of time evolution of turbulence field
  - Application to accretion disk systems

What we want to find out (“Ambition” part)

- Spectrum of Magneto-Rotational Instability (MRI) turbulence
  - MRI in accretion disks has a broad injection scale → inertial range has not resolved
  - Kawazura & Kimura (2024): First time ever to resolve from MHD scale to inertial range
  - At much smaller scales, properties are revealed by reduced MHD (Kawazura et al. 2022)
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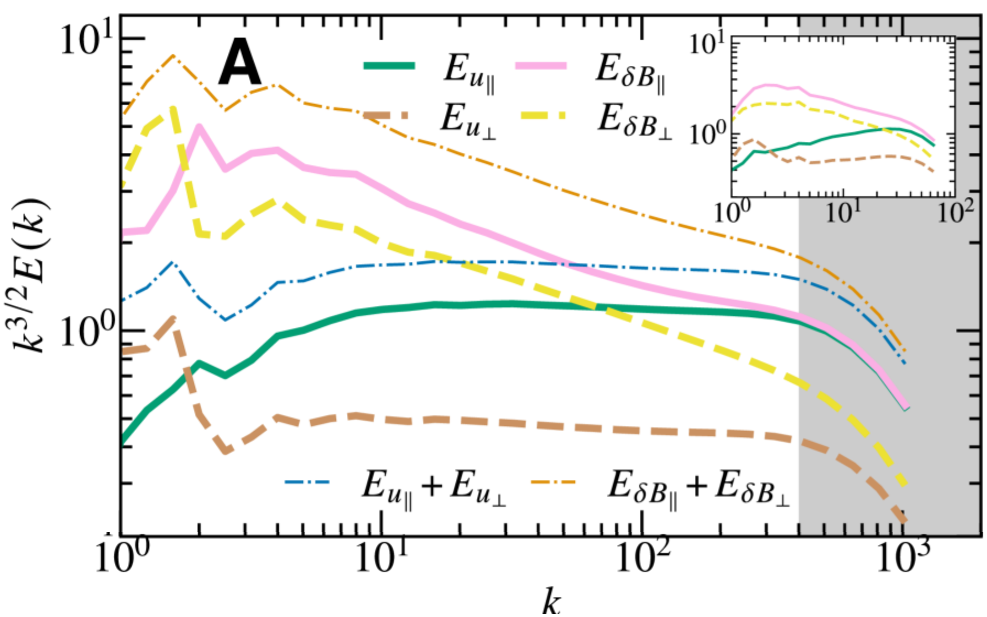

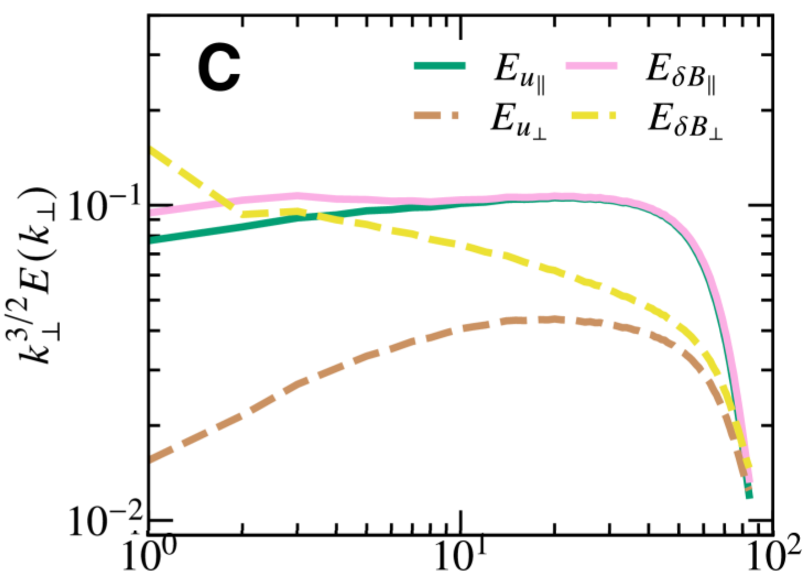

(Kawazura & Kimura (2024))

Basic equation and methodology

- **Fokker-Planck equation**: describing cosmic ray propagation and acceleration

$$\frac{\partial f}{\partial t} = \nabla \cdot (\boldsymbol{\kappa} \cdot \nabla f - \mathbf{V} f) + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(\frac{1}{3}(\nabla \cdot \mathbf{V})p + \Lambda_{\mathrm{loss}}\right)f\right] + \frac{1}{p^2}\frac{\partial}{\partial p}\left(D_{pp}p^2\frac{\partial f}{\partial p}\right)$$

3D MHD simulation → **Fokker-Planck Solver** → **Observable (γ, neutrino, …)**

This work → Development of a post-process code to solve this equation based on MHD simulation in 3D (spatial) + 1D (energy)

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Advection: velocity field (MHD sim.)

Adiabatic heating/loss: velocity field (MHD sim.)

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Cooling: appropriate processes
(e.g., photomeson production)

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$$r_c = \frac{E}{eB}$$

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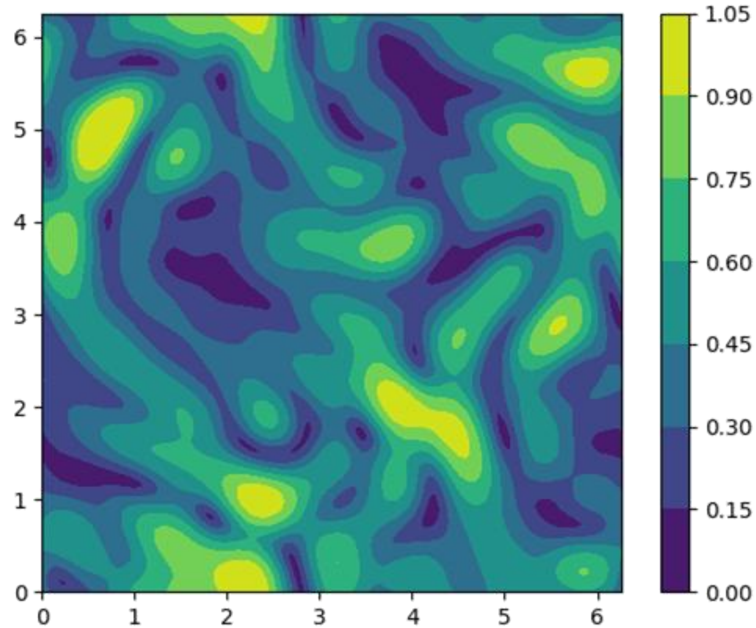
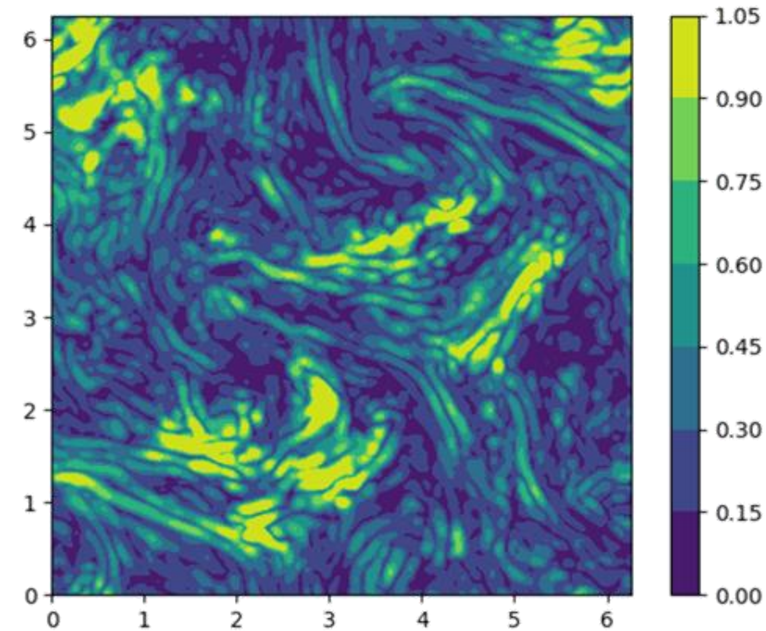
Momentum Diffusion: modeled from MHD simulation

Turbulence felt by high energy particles



Turbulence felt by low energy particles



Y. Kawazura (see also Kawazura & Kimura (2024))

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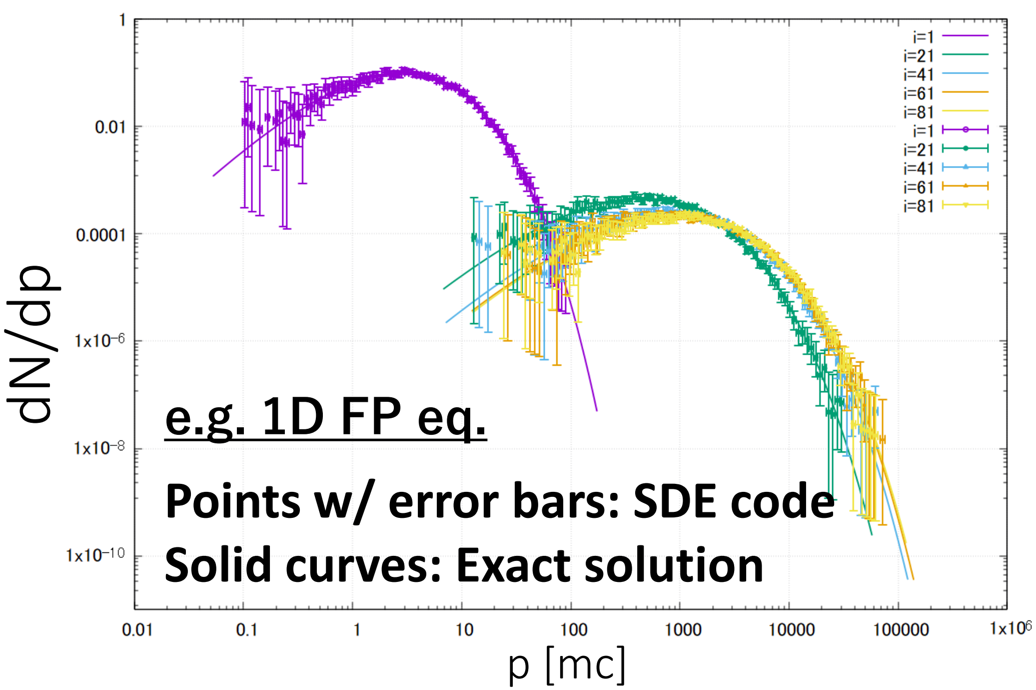
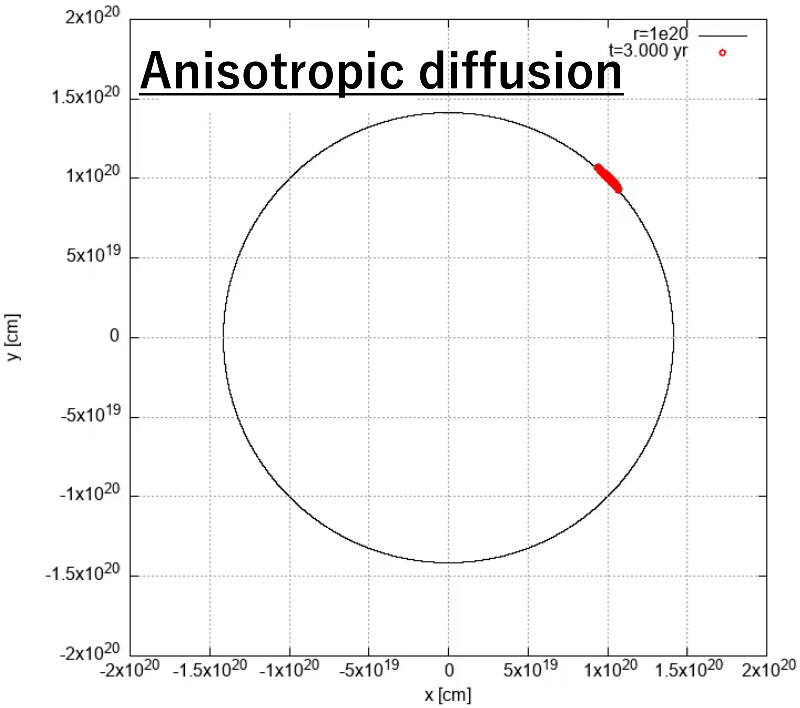
Method & Current status

- Stochastic Differential Equation method (SDE method)
  - Convert partial differential eq. into "many ordinary differential eqs. w/ stochastic terms"
  - Parallelization ◎, Multi-D/species ◎, Stability ◎, Accuracy △

    (∵ Accuracy is determined by statistics→ Can be covered by an efficient computation!)

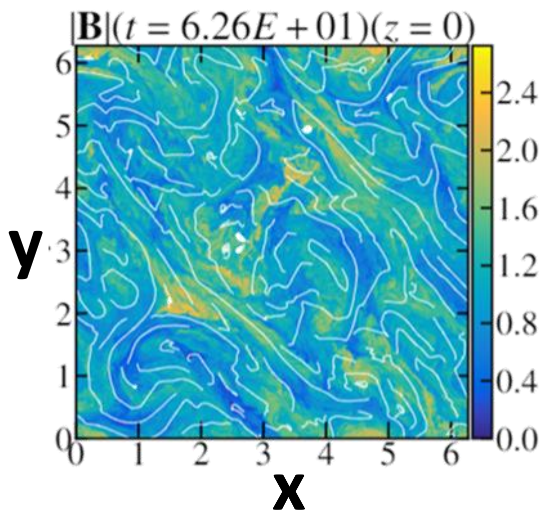
- Current status: Fokker-Planck equation solver part completed!

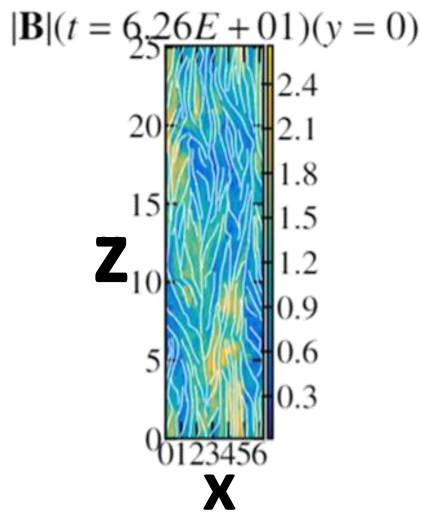




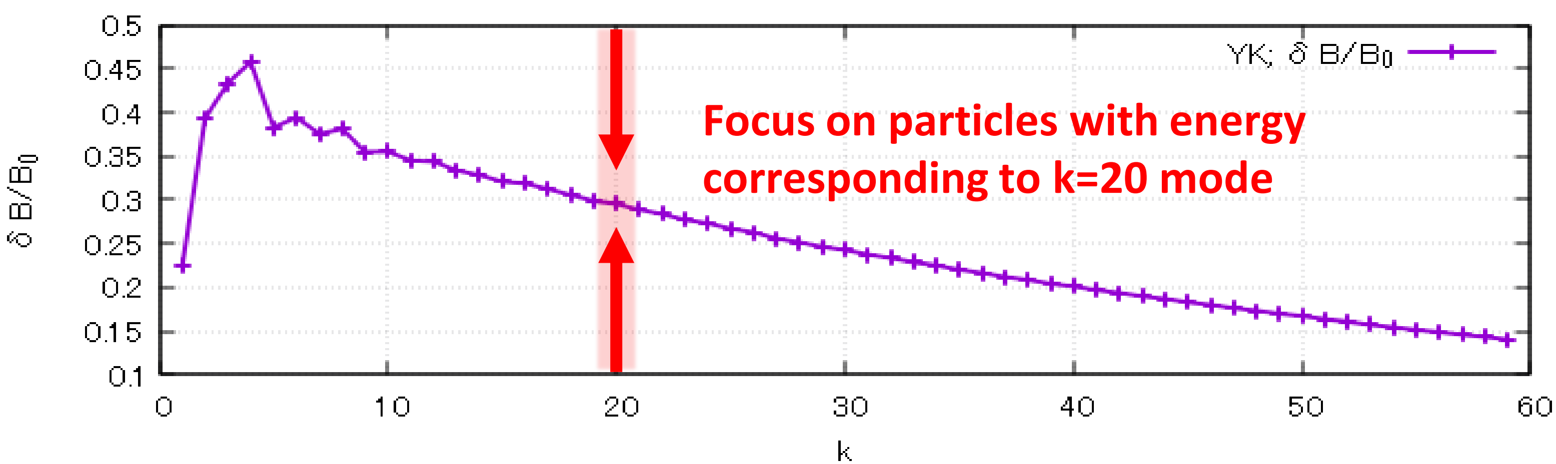
Test Calculation: MHD turbulence

- MHD turbulence (incompressive) (by Y. Kawazura, see also Kawazura & Kimura (2024))
  - Calculate spatial diffusion with MHD simulation data of artificially excited turbulence





- 3D calc. (256x256x256)
- B-field is roughly in the z direction
- Periodic boundary
- Velocity field is ignored (for simplicity)



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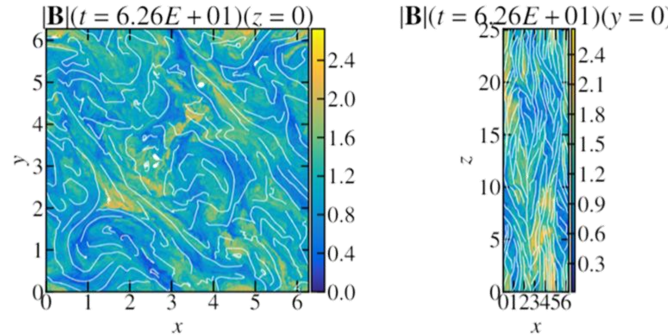

- Model:
  - Quasi-Linear Theory (QLT) + Alfvenic turbulence (e.g., Blandford & Eichler (1987))

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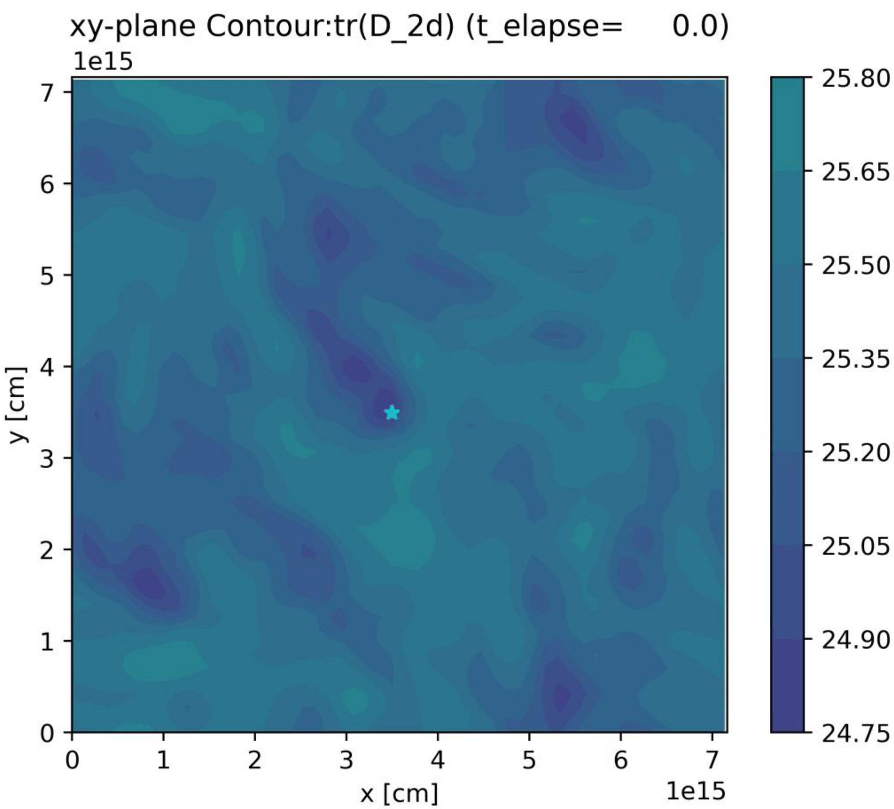
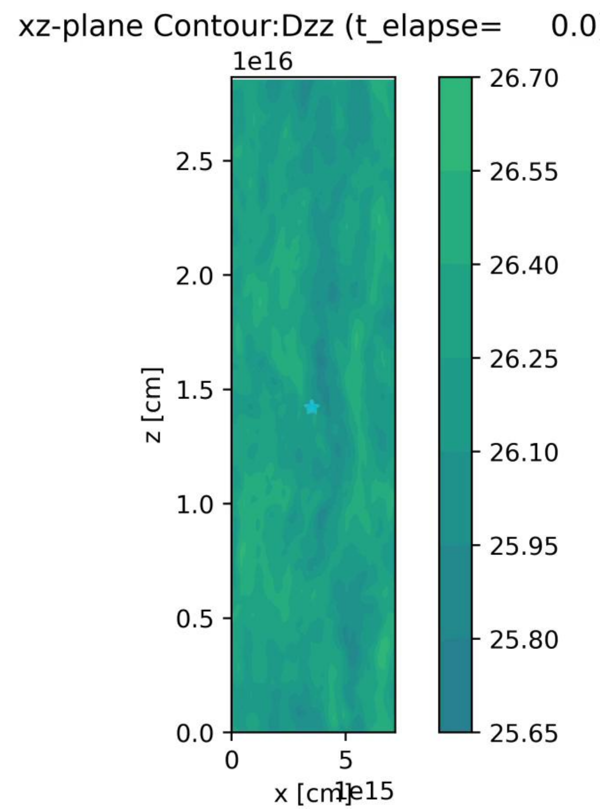
Gyro radius: $r_{\rm L} = \dfrac{E}{eB_0}$





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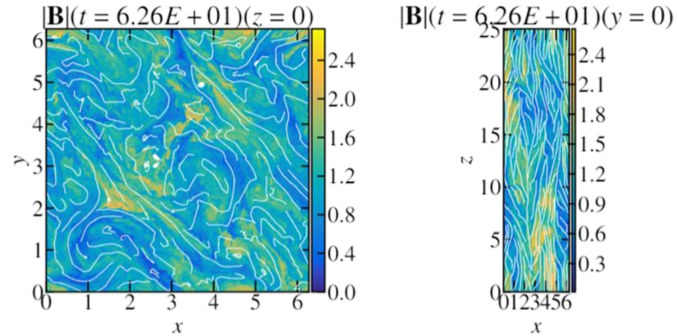

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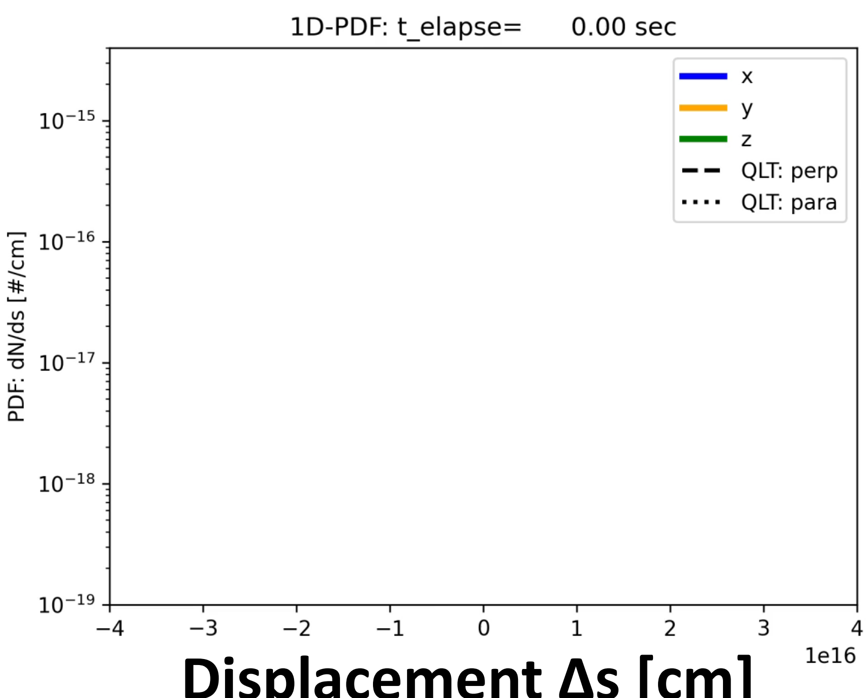
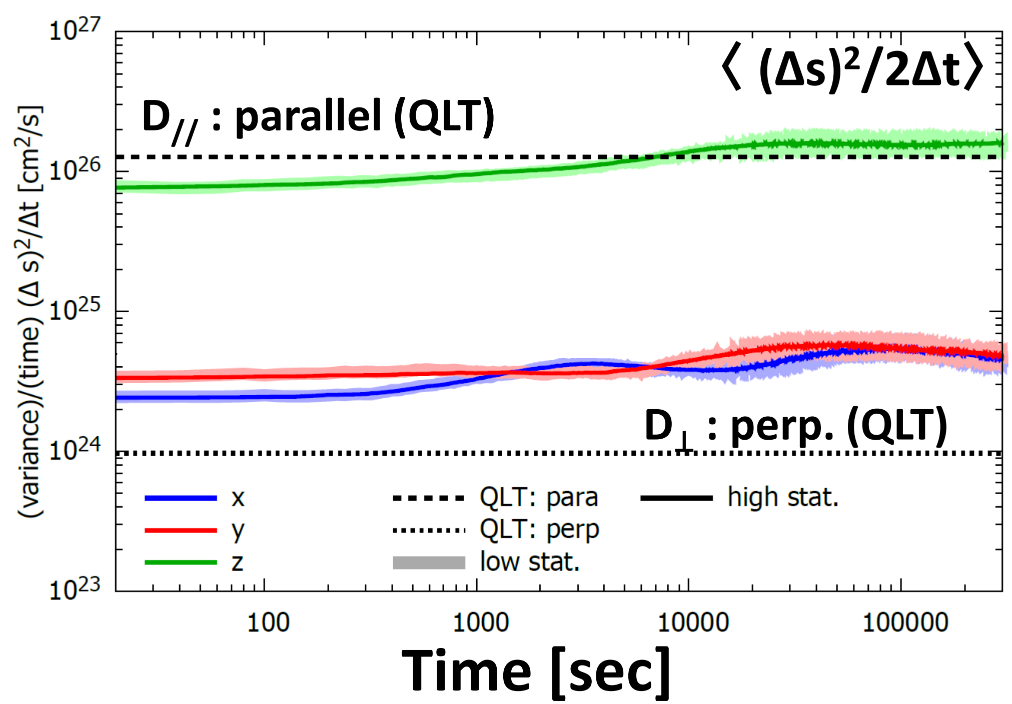
$$D_{\parallel} = \frac{1}{3} r_{\mathrm{L}} c \left(\frac{\delta B}{B_0} \right)^{-2}, \quad D_{\perp} = D_{\parallel} \left(\frac{\delta B}{B_0} \right)^{4}$$

Gyro radius: $r_{\mathrm{L}} = \dfrac{E}{eB_0}$

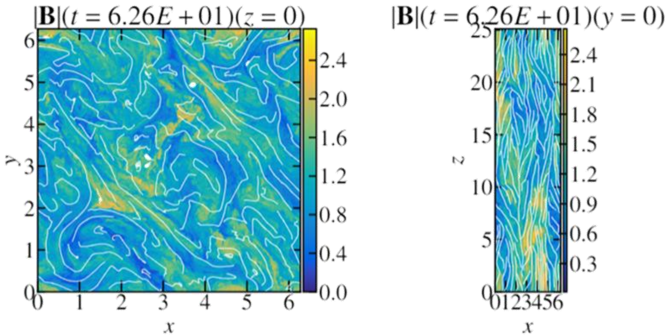
Parallel (z): consistent with QLT

Perpendicular (x,y): much faster than QLT...





Test Calculation: MHD turbulence



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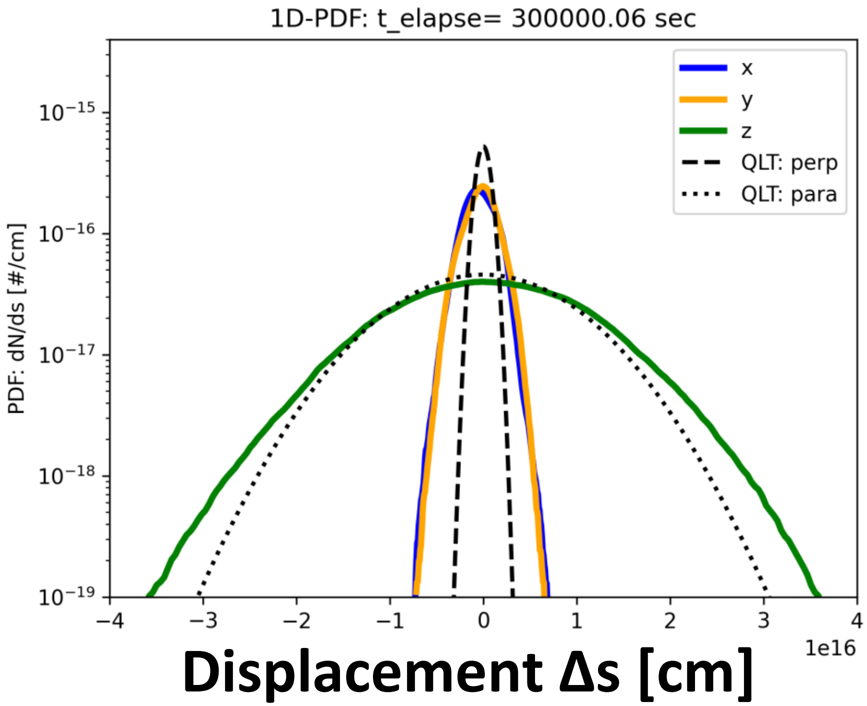

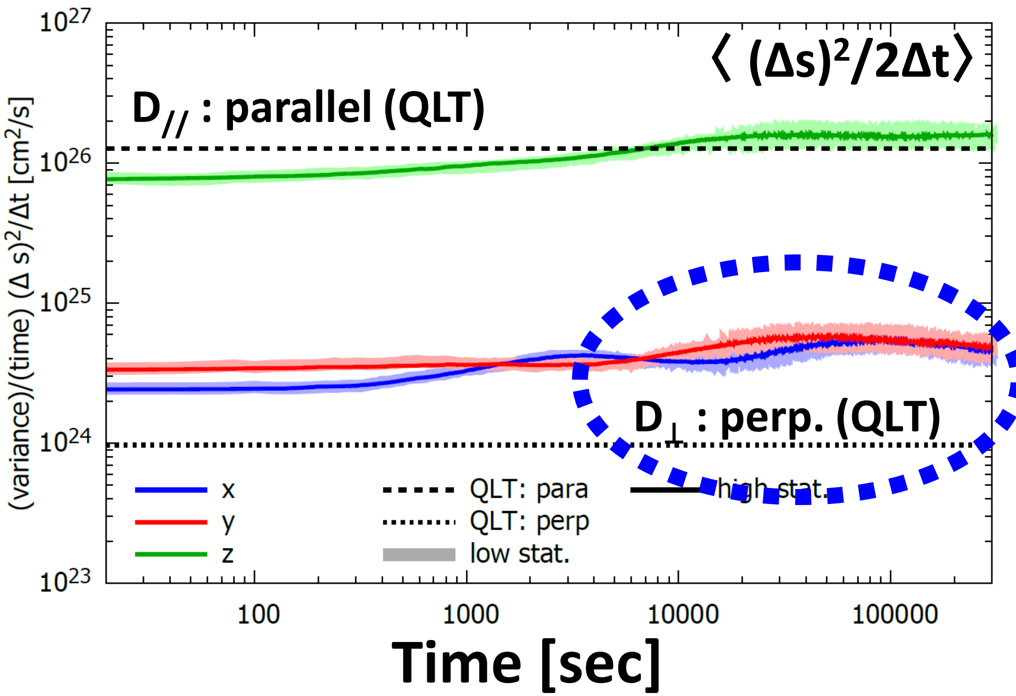

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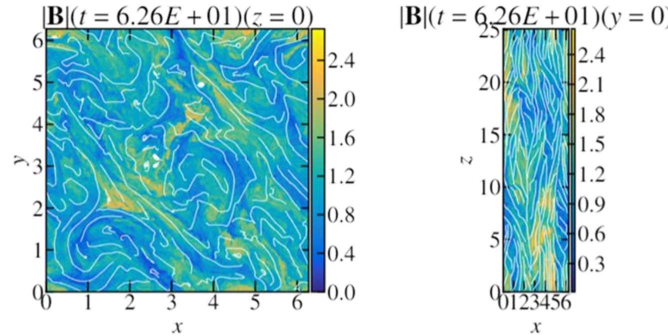

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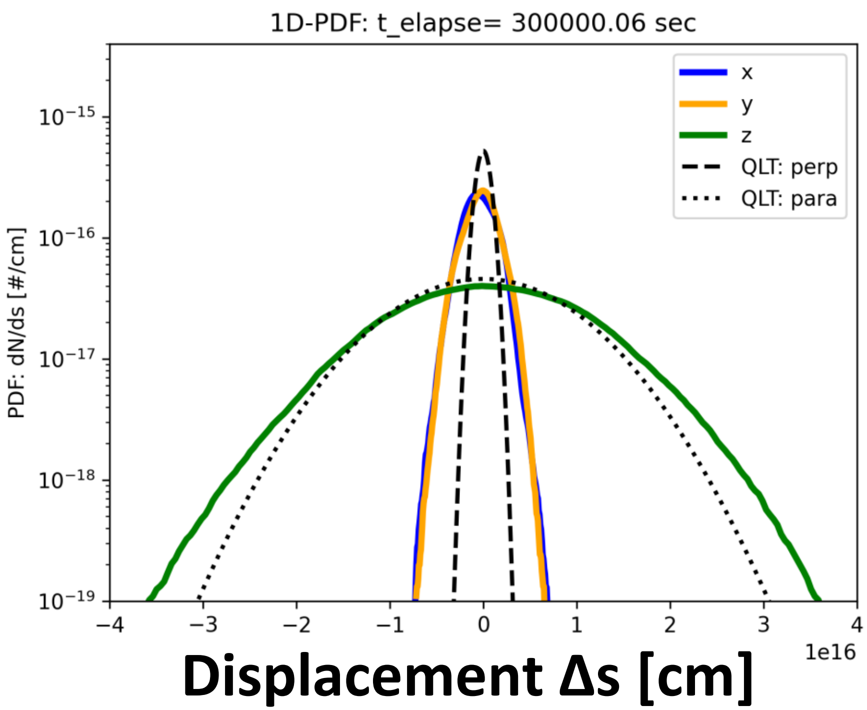
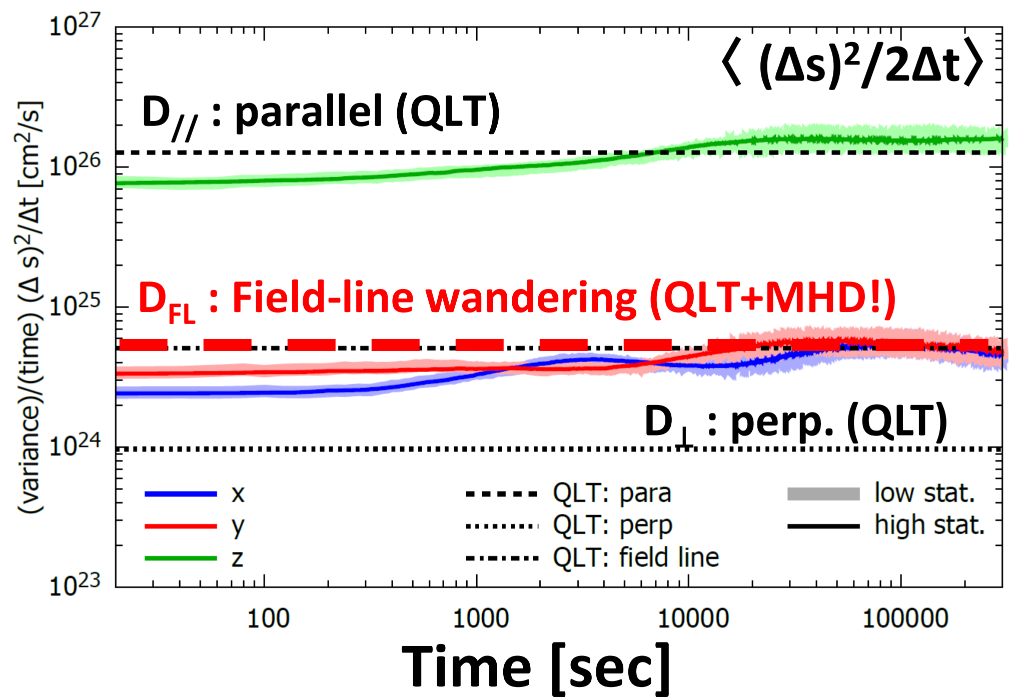
Gyro radius: $r_{\mathrm{L}} = \dfrac{E}{eB_0}$

Perpendicular: consistent with the field-line wandering (MHD effect!)

(e.g., Jokipii & Parker (1968))





Summary / Next steps

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  - We are developing a code to solve for the acceleration and propagation of cosmic rays consistent with MHD simulations
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  - Calculations for multi-energy case (since this work was a mono energy calculation)
  - Calculations including momentum space diffusion (acceleration)
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  - Application to accretion disk systems

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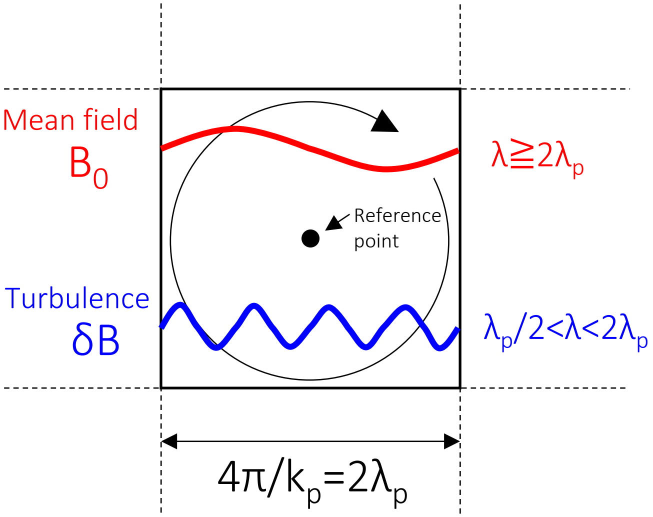
How do we calculate $\delta B/B_0$ at each point & wave number?

- MHD simulation and turbulence
  - MHD simulation only provides the sum of the “mean field (B_0)” and “turbulent field (δB)”.
  - For a given length scale, we need to divide the magnetic field into long-wavelength and short-wavelength components.
- Calculation method
  - First, specify a wave number $k_p=2\pi/\lambda_p$ of the turbulence
  - Take a patch of size $2\lambda_p$ around a reference point
  - Fourier transform of MHD simulation data for each patch
  - The average field and turbulent field are separated
  - The turbulence degree is evaluated from the power ratio
  - Calculate this for each spatial point and the wave number



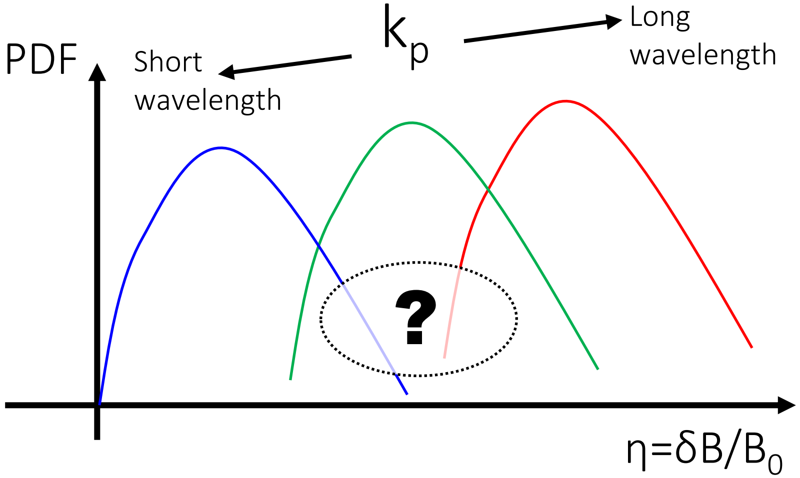
Then we can obtain $\delta B/B_0$ as a function of space and wave number!

Sub-grid modeling

Gyro radius

$$r_L = \frac{E}{eB_0} \propto E$$

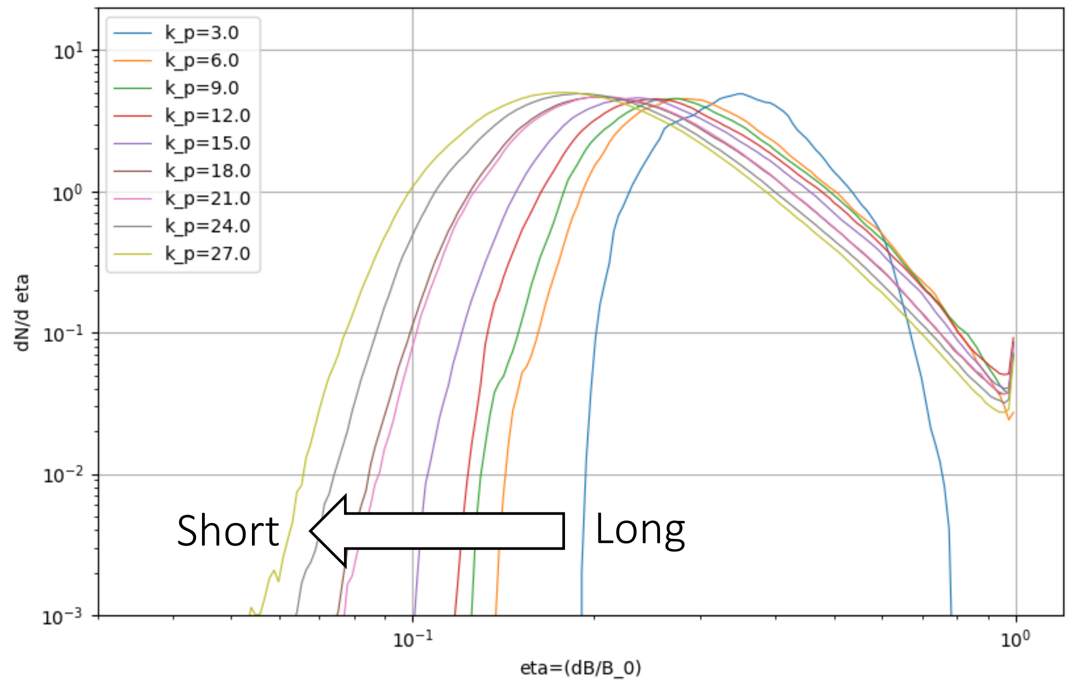
- Wavenumber of turbulence and particle energy
  - Charged particles interact with turbulence of wavelengths around their gyro radius
  - Information on short-wavelength turbulence is necessary to solve the acceleration and propagation of low-energy charged particles
  - However, there are limits to the resolution of MHD calculations, and short-wavelength information must be modeled
- Sub-grid modeling
  - Modeled through the probability distribution function of turbulence
  - The mean field can be roughly obtained from MHD simulation

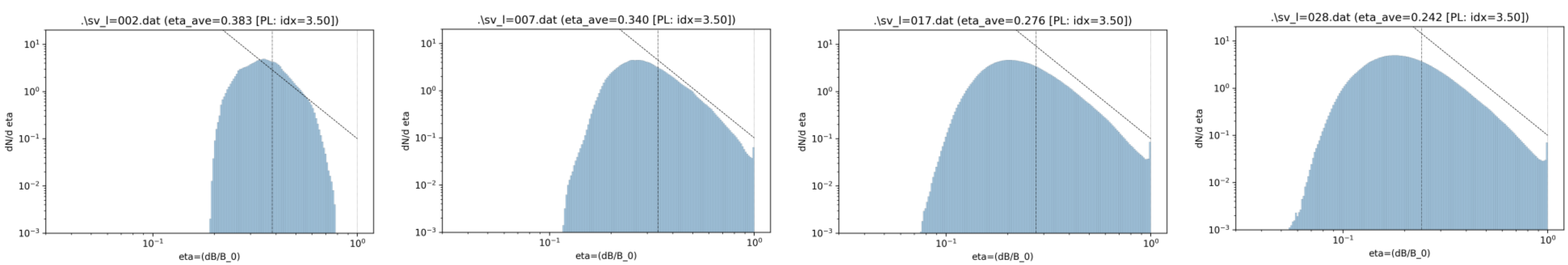


Is it possible to model probability distribution functions of η?

Probability Distribution Function (PDF) of $\eta=\delta B/B_0$

- PDF of $\eta=\delta B/B_0$
  - Analysis of box simulation
  - The PDF appears to converge as the wavelength becomes shorter
  - The peak of PDFs follows the average value for the entire box
  - This universality may be due to the intermittency of turbulence



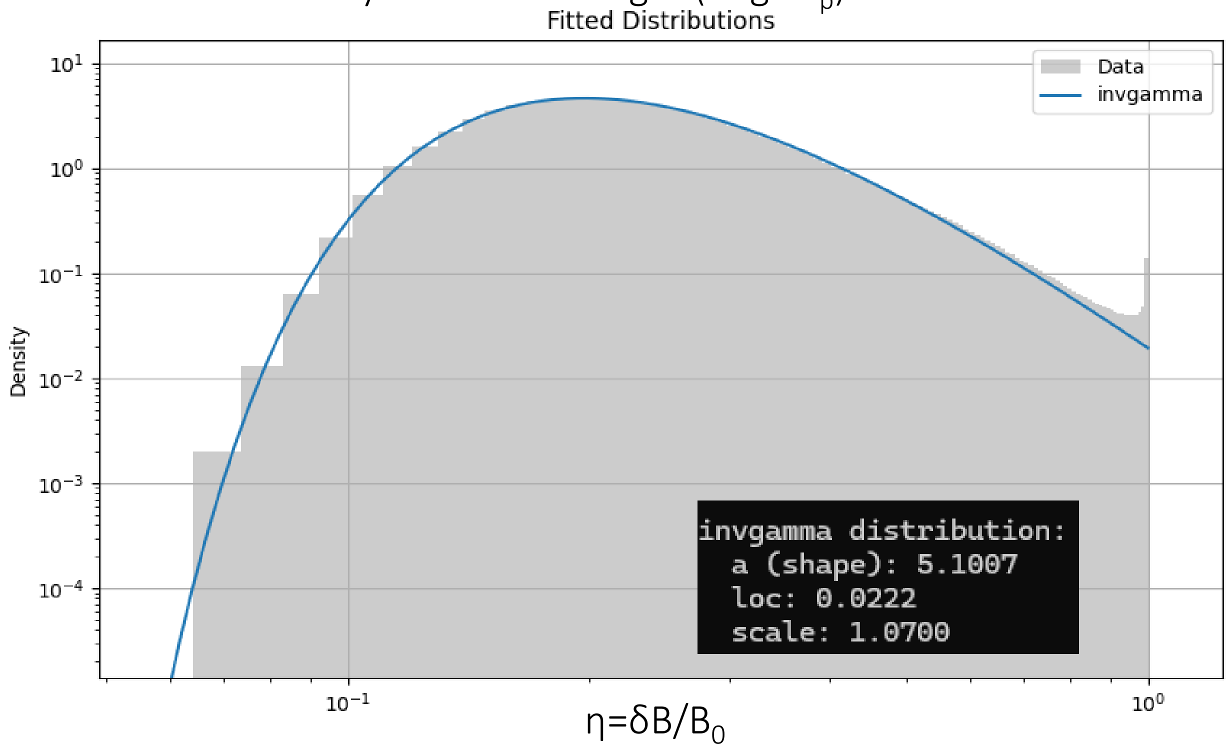


Long wavelength ⟷ Short wavelength

Fitting function

- PDFs are well fitted by the “inverse gamma distribution”
  - β can be obtained from the average value of the box
  - (Even below resolution) predictable from statistical theory of turbulence
  - α is universal for sufficiently short wavelength (large k_p)

$$\boxed{f(x;\alpha,\beta)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\frac{e^{-\beta/x}}{x^{\alpha+1}}}$$



We can use this functional form to model the (much) short wavelength!

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Relationship to other studies in C01

- Kimura-san said…

- MHD Simulation + Test Particle Simulation
  - Solve orbits of CR particles using MHD data sets
  - Enable us to obtain diffusion coefficients
  - limited to CRs with $r_L > \Delta x$

SSK et al. 2016, 2019, in prep

Model for diffusion coefficient

- MHD Simulation + CR Transport simulation
  - Solve CR transport equation using MHD data sets
  - We need a model for diffusion coefficients
  - We can obtain useful info for CRs with $r_L < \Delta x$

Talk by Ishizaki-san; Poster by Kawashima-san

Stochastic Differential Equation (SDE) method

- (Ito-type) stochastic differential equation (SDE)
  - Ordinary differential equation w/ stochastic term
  - Ito-SDE has a following standard form:

$$\frac{d\hat{v}}{dt} = -a(\hat{v}) + b(\hat{v}) \cdot \hat{\xi}$$

$$\Leftrightarrow \; d\hat{v} = -a(\hat{v})dt + b(\hat{v}) \cdot d\hat{L} \Longleftrightarrow \hat{v}(t) = \hat{v}(0) - \int_0^t a(\hat{v}(s))ds + \int_0^t b(\hat{v}(s))d\hat{L}(s)$$

Where a(v) and b(v) are smooth functions, ξ is a stochastic variable generated by Gaussian process

- One-to-one correspondence between a SDE and a PDE (partial differential equation)
  - The ensemble of solutions to the SDE follows a PDE called the master equation
  - In particular, for the Ito-SDE, the master equation is the diffusion-advection equation

$$\frac{d\hat{v}}{dt} = \underbrace{-a(\hat{v})}_{\textbf{Drift}} + \underbrace{b(\hat{v})}_{\substack{\textbf{Random walk} \\ \textbf{(mean free path)}}} \cdot \hat{\xi} \;\Longleftrightarrow\; \frac{\partial P(v,t)}{\partial t} = \frac{\partial}{\partial v}(\underbrace{a(v)}_{\textbf{Advection}} P(v,t)) + \frac{\partial^2}{\partial v^2}\left(\underbrace{\frac{1}{2}b(v)^2}_{\textbf{Diffusion}} P(v,t)\right)$$

$$\left(\langle f(v) \rangle = \int f(v) P(v,t) dv\right)$$

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- One-to-one correspondence between a SDE and a PDE (partial differential equation)

The advection-diffusion equation can be solved by solving a large number of Ito-SDEs and taking their ensemble!

$$\frac{d\hat{v}}{dt} = \underbrace{-a(\hat{v})}_{\text{Drift}} + \underbrace{b(\hat{v}) \cdot \hat{\xi}}_{\text{Random walk (mean free path)}} \Longleftrightarrow \frac{\partial P(v,t)}{\partial t} = \frac{\partial}{\partial v}\underbrace{(a(v)P(v,t))}_{\text{Advection}} + \frac{\partial^2}{\partial v^2}\underbrace{\left(\frac{1}{2}b(v)^2 P(v,t)\right)}_{\text{Diffusion}}$$

$$\left(\langle f(v)\rangle = \int f(v)P(v,t)dv\right)$$

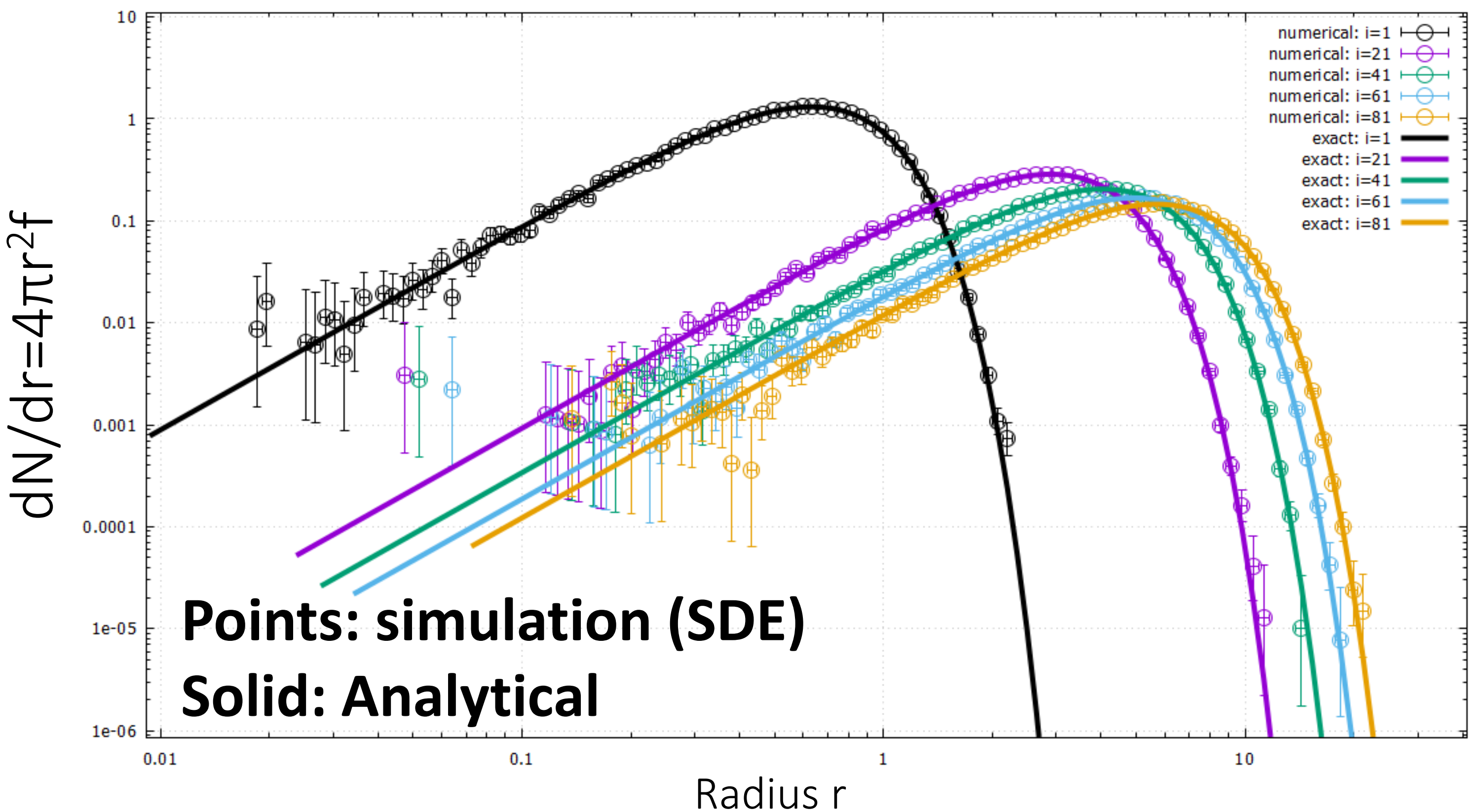
Advantages / Disadvantages (vs. grid-based method)

- Advantages
  - Easily expandable to higher dimensions and multi-particle species
  - High parallelization efficiency ~100% (∵ just solve many independent ODEs)
  - Computational stability is easily ensured because CFL conditions caused by grid size do not occur
  - Intuitive introduction of new effects, since only effects over single particle equations are considered
- Disadvantages
  - Difficult to set boundary conditions
    - But, in our field, we basically consider relatively simple boundary conditions (e.g., "0" at infinity)
  - Computational accuracy depends on particle number statistics
    - Can be compensated by high parallelization efficiency

Test-calculation: simple diffusion in 3D-space

- 3D diffusion: D=1.0, impulsive injection in t=0 (@r_0=0)
  - The calculation is performed in Cartesian coordinate (x,y,z)

$$\frac{\partial f}{\partial t} - D\Delta f = 0 \qquad G(\boldsymbol{r}, t; \boldsymbol{r}_0, t_0) = \frac{1}{\pi^{3/2}\left[4D\left(t-t_0\right)\right]^{3/2}} \exp\left(-\frac{\left|\boldsymbol{r}-\boldsymbol{r}_0\right|^2}{4D\left(t-t_0\right)}\right)$$



Test-Calculation 2: Stochastic acceleration

- Mertsch 2011; Green's function of the FP equation in momentum space

$$\frac{\partial f(p,t)}{\partial t} = -\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2\left(-D_{pp}(p,t)\frac{\partial f(p,t)}{\partial p} + A(p,t)f(p,t)\right)\right)$$

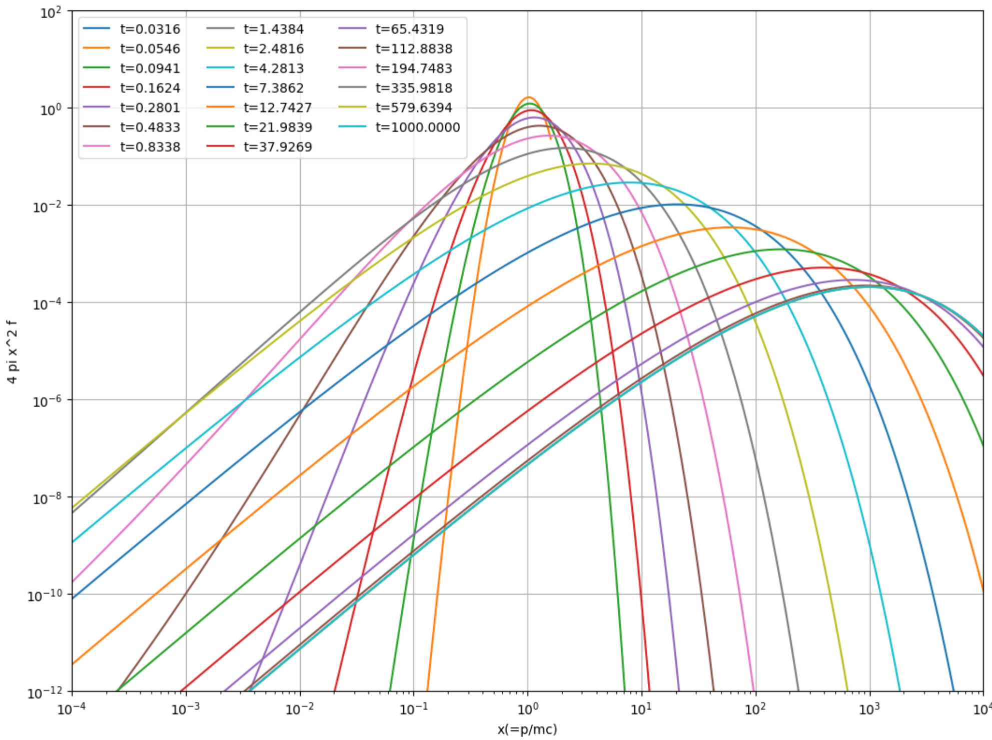
$$A(p,t) = mc\left(\frac{p}{mc}\right)a_0, \quad D_{pp}(p,t) = k_0(mc)^2\left(\frac{p}{mc}\right)^q$$

Injcted at $t=t_0$ and $x=x_0$ ($p=x_0mc$)

$$f = \frac{1}{(mc)^3 4\pi x_0^3}\exp\left(-\frac{3}{2}a_0(t-t_0)\right)\frac{a_0}{k_0}\frac{(xx_0)^{(2-q)/2}\sqrt{g(t)}}{1-g(t)}\exp\left(-\frac{a_0}{(2-q)k_0}\frac{x^{2-q}g(t)+x_0^{2-q}}{1-g(t)}\right)I_{\frac{1+q}{2-q}}\left[\frac{a_0}{(2-q)k_0}\frac{2(xx_0)^{(2-q)/2}\sqrt{g(t)}}{1-g(t)}\right]\left(\frac{x}{x_0}\right)^{-3/2}$$

$$\left(g(t) = \exp\left[-(2-q)a_0(t-t_0)\right]\right)$$

I_ν: modified Bessel function



$m=c=1$
$x_0=1$
$t_0=0$
$a_0=-0.1$ **(<0)**
$k_0=0.5$
$q=5/3$

Steady state at t~>100

Test-Calculation 2: Stochastic Acceleration

- Example of formulation in SDE
  - If ϕ=4πp²f, we can rewrite the FP equation in the form of the master equation for an Ito-SDE

$$\frac{\partial f(p,t)}{\partial t} = -\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2\left(-D_{pp}(p,t)\frac{\partial f(p,t)}{\partial p} + A(p,t)f(p,t)\right)\right)$$

$$\Longrightarrow \quad \frac{\partial \phi(p,t)}{\partial t} = -\frac{\partial}{\partial p}\left(\left(A(p,t) + \frac{2D_{pp}(p,t)}{p} + \frac{\partial D_{pp}(p,t)}{\partial p}\right)\phi(p,t)\right) + \frac{\partial^2}{\partial p^2}(D_{pp}(p,t)\phi(p,t))$$

- Ito-type SDEs and Master Equations (Restated)

$$\frac{d\hat{v}}{dt} = -a(\hat{v}) + b(\hat{v})\cdot\hat{\xi} \quad \leftrightarrow \quad \frac{\partial P(v,t)}{\partial t} = \frac{\partial}{\partial v}(a(v)P(v,t)) + \frac{\partial^2}{\partial v^2}\left(\frac{1}{2}b(v)^2P(v,t)\right)$$

- By comparing the coefficients, the SDE corresponding to the FP equation is obtained as:

$$d\hat{p} = \left(A(p,t) + \frac{2D_{pp}(p,t)}{p} + \frac{\partial D_{pp}(p,t)}{\partial p}\right)dt + \sqrt{2D_{pp}(p,t)}d\hat{W}$$

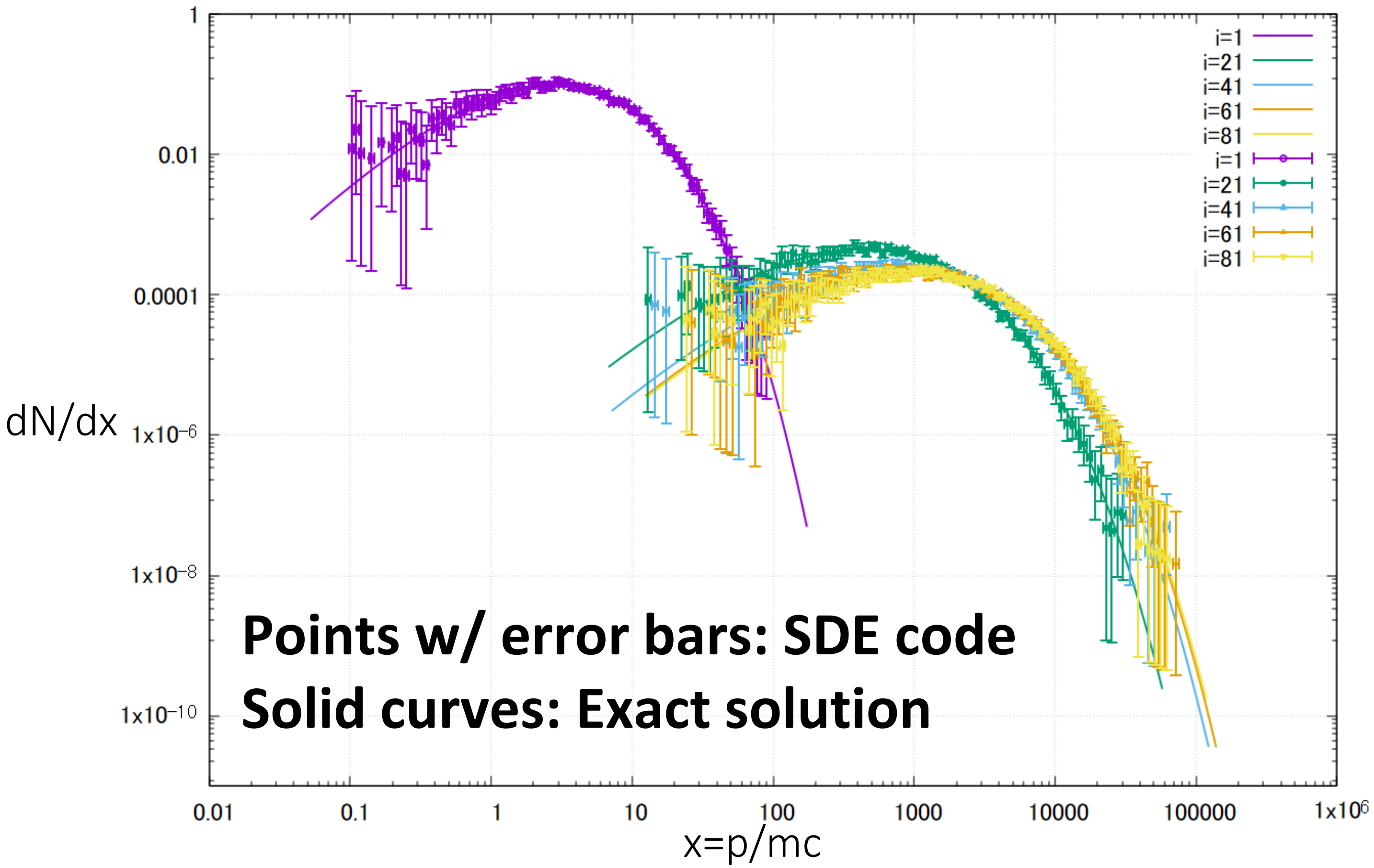
Test-Calculation 2: Stochastic Acceleration

- Comparison with exact solution

$$d\hat{p} = \left(A(p,t) + \frac{2D_{pp}(p,t)}{p} + \frac{\partial D_{pp}(p,t)}{\partial p} \right) dt + \sqrt{2D_{pp}(p,t)} d\hat{W}$$

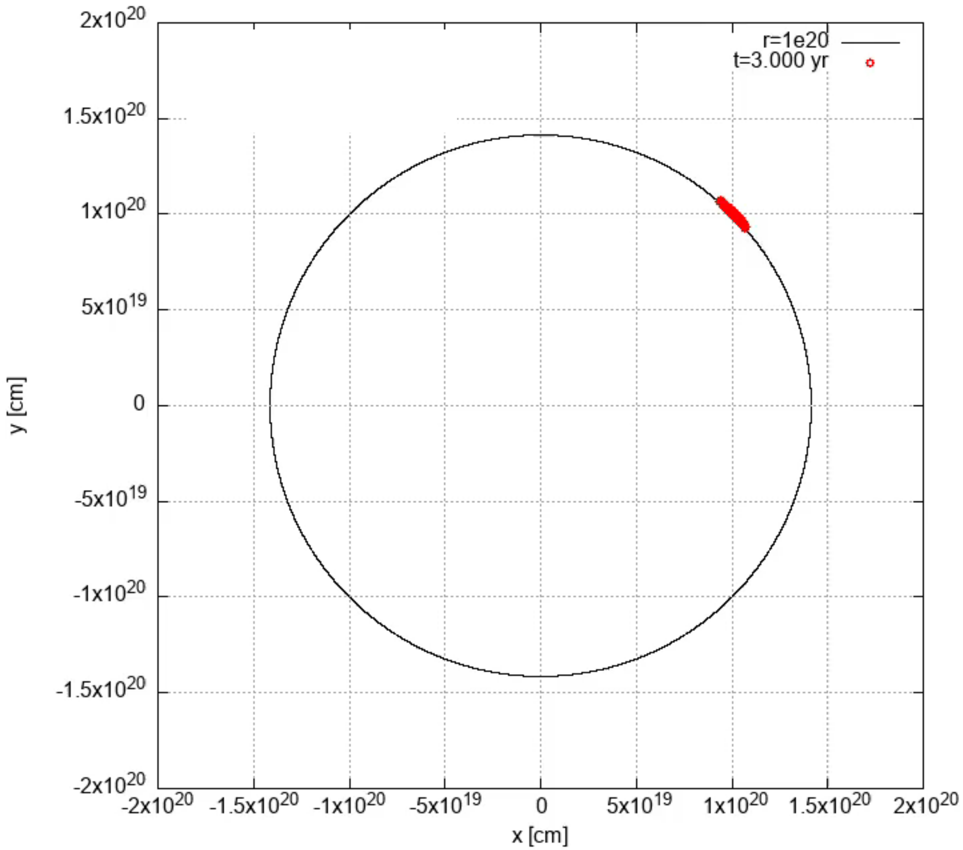
  - Points w/ error bars: simulation by SDE code, Solid line: analytical solution

Successfully solve the FP equation with this technique!



Test-Calculation 3: anisotropic diffusion

- Magnetic field: ϕ-direction
  - $\kappa_\perp = 10^{-6}\kappa_{//}$, $\kappa_{//} = 3.0 \times 10^{28}(E/10\ \mathrm{GeV})^{1/3}$
  - Setting: 1000-particles, 3000yr, Power-law injection to energy space



$$\overleftrightarrow{\kappa} = \begin{pmatrix} \kappa_\perp + (\kappa_\parallel - \kappa_\perp)\dfrac{y^2}{x^2+y^2} & -(\kappa_\parallel - \kappa_\perp)\dfrac{xy}{x^2+y^2} & 0 \\ -(\kappa_\parallel - \kappa_\perp)\dfrac{yx}{x^2+y^2} & \kappa_\perp + (\kappa_\parallel - \kappa_\perp)\dfrac{x^2}{x^2+y^2} & 0 \\ 0 & 0 & \kappa_\perp \end{pmatrix}$$

$$\nabla \cdot \overleftrightarrow{\kappa} = -\frac{\kappa_\parallel - \kappa_\perp}{\rho}\hat{\rho}$$

$$(\rho = \sqrt{x^2+y^2})$$

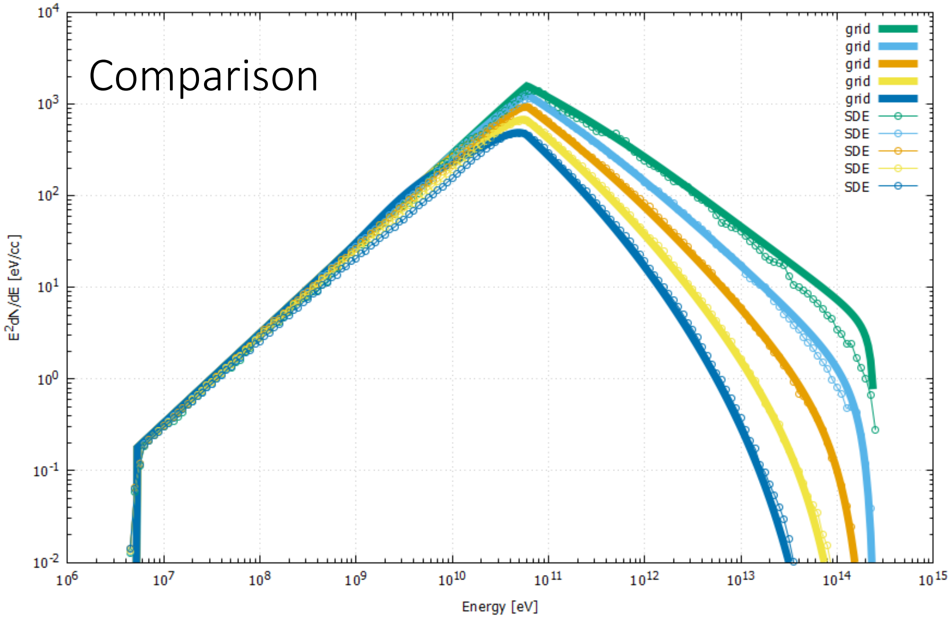
Anisotropic diffusion is solved well!

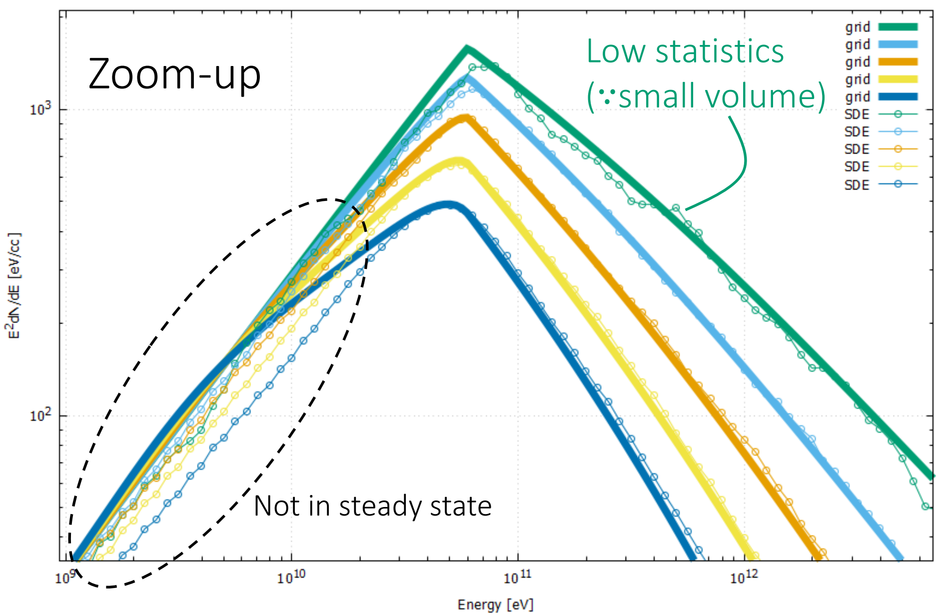
Now, Roughly all processes have been introduced

⇒Next, connection with MHD calculation

Test-Calculation 4: Pulsar Wind Nebulae

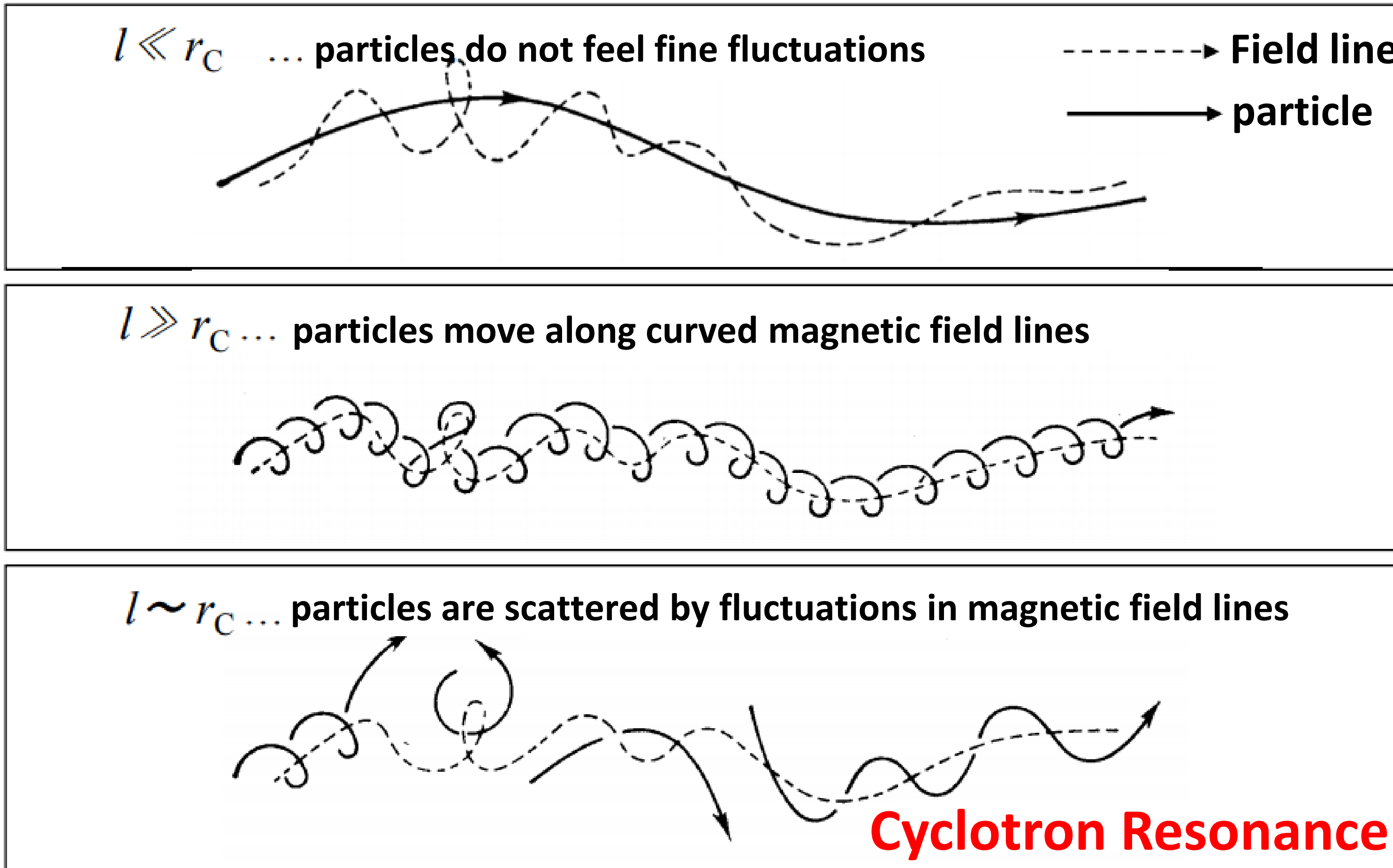
- MHD
  - By loading (mock) 3DMHD data, interpolation function of velocity and magnetic fields are generated
  - Comparison: Spherically symmetric steady-state diffusion model of PWNe (Ishizaki+2018; left)
  - Differences in calculation setup:
    - Boundary: SDE code injects particles multiple times at appropriate time intervals to reproduce fixed boundaries
    - Grid code solves the steady state eq., while SDE code solves the time-dependent eq. until it becomes steady.
  - Energy spectrum of particles at each radius (calculated in 3+1 dimensions in the SDE code)





Interaction between charged particles and turbulence

r_c (gyro radius) $\longleftrightarrow$ l : wavelengths of magnetic field disturbance



Terasawa-san's slide translated and modified by WI (2011/7/29　タウンミーティング「地下素粒子実験・宇宙観測」)